INTERNATIONAL CENTRE FOR ECONOMIC RESEARCH

ICER

WORKING PAPER SERIES

Massimo Marinacci and Luigi Montrucchio

INTRODUCTION TO THE MATHEMATICS OF AMBIGUITY

Working Paper no. 34/2003
Novembre 2003

APPLIED MATHEMATICS
WORKING PAPER SERIES

Introduction to the Mathematics of Ambiguity*

Massimo Marinacci and Luigi Montrucchio
Dipartimento di Statistica e Matematica Applicata and ICER
Università di Torino†

Contents

*Draft of October 2003. To appear in Itzhak Gilboa "Uncertainty in economic theory: a collection of essays in honor of David Schmeidler's 65^{th} birthday", Routledge, New York. We thank Fabio Maccheroni for his very insightful suggestions, which greatly improved this chapter. The financial support of MIUR (Ministero dell'Istruzione, Università e Ricerca Scientifica) is gratefully acknowledged.

†Piazza Arbarello 8, 10122 Torino, Italy. E-mails: massimo.marinacci@unito.it and luigi.montrucchio@unito.it; URL: http://web.econ.unito.it/gma.

1 Introduction

As discussed at length in chapters 1-3, some mathematical objects play a central role in Schmeidler's decision-theoretic ideas. In this chapter we provide some more details on them.

One of the novelties of Schmeidler's decision theory papers was the use of general set functions, not necessarily additive, to model "ambiguous" beliefs. This provided a new and intriguing motivation for the study of these mathematical objects, already studied from a different standpoint in cooperative game theory, another field where David Schmeidler has made important contributions.

Here we overview the main properties of such set functions. Most of the results we will present are known, though often not in the generality in which we state and prove them. In the attempt to provide streamlined proofs and more general statements, we sometimes came up with novel arguments.

2 Set Functions

2.1 Basic Properties

We begin by studying the basic properties of set functions. We use the setting of cooperative game theory as most of these concepts originated there; their decision-theoretic interpretation is treated in great detail in chapters 1-3 and 13, as well as in many of the articles collected in this book.

Let Ω be a set of players and Σ an algebra of admissible coalitions in Ω. A (transferable utility) game is a real-valued set function $\nu : \Sigma \to \mathbb{R}$ with the only requirement that $\nu(\varnothing) = 0$. Given a coalition $A \in \Sigma$, the number

$\nu(A)$ is interpreted as its worth, that is, the overall value that his members can achieve by teaming up.

The condition $\nu(\varnothing) = 0$ reflects the obvious fact that the worth of the empty coalition is zero; a priori, nothing more is assumed in defining a game ν. In the game theory literature several additional conditions have been considered. In particular, a game ν is[1]

1. *positive* if $\nu(A) \geq 0$ for all A;
2. *bounded* if $\sup_{A\in\Sigma} |\nu(A)| < +\infty$;
3. *monotone* if $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$;
4. *superadditive* if $\nu(A \cup B) \geq \nu(A) + \nu(B)$ for all pairwise disjoint sets A and B;
5. *convex* (*supermodular*) if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all A, B;
6. *additive* (a *charge*) if $\nu(A \cup B) = \nu(A) + \nu(B)$ for all pairwise disjoint sets A and B.

All these conditions have natural game-theoretic interpretations (see, e.g., Moulin [44] and Owen [48]). For example, a game is monotone when larger coalitions can achieve higher values, and it is superadditive when combining disjoint coalitions results in more than proportional increases in value. As to supermodularity, it is a stronger property than superadditivity and it can be equivalently formulated as

$$\nu(B \cup C \cup A) - \nu(B \cup C) \geq \nu(B \cup A) - \nu(B), \tag{1}$$

for all disjoint sets A, B, and C; hence, it can be interpreted as a property of increasing marginal values (see Proposition 34 below).

Some assumptions of a more technical nature are also often assumed. For example, a game ν is:

7. *outer* (*inner*, resp.) *continuous at* A if $\lim_{n\to\infty} \nu(A_n) = \nu(A)$ whenever $A_n \downarrow A$ ($A_n \uparrow A$, resp.);

[1] In the sequel subsets of Ω are understood to be in Σ even where not stated explicitly and they are referred to both as sets and as coalitions.

8. *continuous at* A if it is both inner and outer continuous at A;

9. *continuous* if it is continuous at each A;

10. *countably additive* (a *measure*) if $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu\left(A_i\right)$ for all countable collections of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

We get important classes of games by combining some of the previous properties. In particular, monotone games are called *capacities*, additive games are called *charges*, and countably additive games are called *measures*. Finally, positive games ν that are normalized with $\nu\left(\Omega\right) = 1$ are called *probabilities*. Notice that capacities are always positive and bounded, while positive superadditive games are always capacities.

Given a charge μ, its *total variation norm* $\|\mu\|$ is given by

$$\sup \sum_{i=1}^{n} \left|\mu\left(A_i\right) - \mu\left(A_{i-1}\right)\right|, \tag{2}$$

where the supremum is taken over all finite chains $\varnothing = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = \Omega$. Denote by $ba\left(\Sigma\right)$ and $ca\left(\Sigma\right)$ the vector spaces of all charges and of all measures having finite total variation norm, respectively. By classic results (see, e.g., [20] and [5]), a charge has finite total variation if and only if it is bounded, and both $ba\left(\Sigma\right)$ and $ca\left(\Sigma\right)$ are Banach spaces when endowed with the total variation norm. In particular, $ca\left(\Sigma\right)$ is a closed subspace of $ba\left(\Sigma\right)$.

In view of these classic results, it is natural to wonder whether a useful norm can be introduced in more general spaces of games. Aumann and Shapley [2] showed that this is the case by introducing the variation norm on the space of all games. Given a game ν, its *variation norm* $\|\nu\|$ is given by

$$\sup \sum_{i=1}^{n} \left|\nu\left(A_i\right) - \nu\left(A_{i-1}\right)\right|, \tag{3}$$

where the supremum is taken over all finite chains $\varnothing = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = \Omega$. If ν is a charge, the variation norm $\|\nu\|$ reduces to the total variation norm. Moreover, all finite games are of bounded variation as they have a finite number of finite chains.

Denote by $bv\left(\Sigma\right)$ the vector space of all games ν having finite variation norm. Aumann and Shapley [2] proved the following noteworthy properties.

Proposition 1 *A game belongs to* $bv(\Sigma)$ *if and only if it can be written as the difference of two capacities. Moreover,* $bv(\Sigma)$ *endowed with the variation norm is a Banach space, and* $ba(\Sigma)$ *and* $ca(\Sigma)$ *are closed subspaces of* $bv(\Sigma)$.[2]

In view of this result, we can say that $bv(\Sigma)$ is a Banach environment for not necessarily additive games that generalizes the classic spaces $ba(\Sigma)$ and $ca(\Sigma)$. In the sequel we will mostly consider games belonging to it.

We close this section by observing that each game ν has a dual game $\overline{\nu}$ defined by $\overline{\nu}(A) = \nu(\Omega) - \nu(A^c)$ for each A. From the definition it immediately follows that:

- $\overline{\overline{\nu}} = \nu$;
- ν is monotone if and only if $\overline{\nu}$ does;
- ν belongs to $bv(\Sigma)$ if and only if $\overline{\nu}$ does.

More important, dual games have "dual" properties relative to the original game. For example:

- ν is convex if and only if $\overline{\nu}$ is concave, i.e., $\overline{\nu}(A \cup B) + \overline{\nu}(A \cap B) \leq \overline{\nu}(A) + \overline{\nu}(B)$ for all A, B;
- ν is inner continuous at A if and only if $\overline{\nu}$ is outer continuous at A^c.

For charges μ it clearly holds $\mu = \overline{\mu}$. Without additivity, ν and $\overline{\nu}$ are in general distinct games (see Proposition 4) and sometimes it is useful to consider the pair $(\nu, \overline{\nu})$ rather than only ν.

Example 2 The duality between ν and $\overline{\nu}$ does not hold for all properties. For example, it is false that ν is superadditive if and only if $\overline{\nu}$ is subadditive. Consider the game ν on $\Omega = \{\omega_1, \omega_2, \omega_3\}$ given by $\nu(\omega_i) = 0$ for $i = 1, 2, 3$, $\nu(\omega_i \cup \omega_j) = 5/6$ for $i, j = 1, 2, 3$, and $\nu(\Omega) = 1$. Its dual $\overline{\nu}$ is given by $\overline{\nu}(\omega_i) = 1/6$ for $i = 1, 2, 3$, $\overline{\nu}(\omega_i \cup \omega_j) = 1$ for $i, j = 1, 2, 3$, and $\nu(\Omega) = 1$. While ν is superadditive, its dual is not subadditive. In fact, $\overline{\nu}(\omega_1 \cup \omega_2) = 1 > \overline{\nu}(\omega_1) + \overline{\nu}(\omega_2) = 1/3$. Normalized superadditive games having subadditive duals are sometimes called upper probabilities (see [70] and the references therein contained).

[2] Maccheroni and Ruckle [39] recently proved that $(bv(\Sigma), \|\cdot\|)$ is a dual Banach space.

2.2 The Core

Given a game ν, its core is the (possibly empty) set given by

$$core(\nu) = \{\mu \in ba(\Sigma) : \mu(A) \geq \nu(A) \text{ for each } A \text{ and } \mu(\Omega) = \nu(\Omega)\}.$$

In other words, the core of ν is the set of all suitably normalized charges that setwise dominate ν. Notice that

$$\begin{aligned} core(\nu) &= \{\mu \in ba(\Sigma) : \nu \leq \mu \leq \overline{\nu}\} \\ &= \{\mu \in ba(\Sigma) : \mu(A) \leq \overline{\nu}(A) \text{ for each } A \text{ and } \mu(\Omega) = \nu(\Omega)\}, \end{aligned}$$

and so the core can be also regarded as the set of charges "sandwiched" between the game and its dual, as well as the set of charges setwise dominated by the dual game.

The core is a fundamental solution concept in cooperative game theory, where it is interpreted as the set of undominated allocations (see [44] and [48]). After Schmeidler's seminal works, the core plays an important role in decision theory as well, as detailed in chapters 1-3.

Mathematically, the interest of the core lies in the connection it provides between games and charges, which, unlike games, are familiar objects in measure theory. As it will be seen later, useful properties of games can be deduced via the core from classic properties of charges.

The core is a convex subset of $ba(\Sigma)$. More interestingly, it has the following compactness property.[3]

Proposition 3 *When nonempty, the core of a bounded game is weak*-compact.*

Proof. Let $\mu \in core(\nu)$ and let $k = 2\sup_{A\in\Sigma}|\nu(A)|$. For each A it clearly holds $\mu(A) \geq \nu(A) \geq -k$. On the other hand,

$$\mu(A) = \mu(\Omega) - \mu(A^c) \leq \nu(\Omega) - \nu(A^c) \leq 2\sup_{A\in\Sigma}|\nu(A)|,$$

and so $|\mu(A)| \leq k$. By [20, p. 97], $\|\mu\| \leq 2k$, which implies

$$core(\nu) \subseteq \{\mu \in ba(\Sigma) : \|\mu\| \leq 2k\}.$$

By the Alaoglu Theorem (see [20, p. 424]), $\{\mu \in ba(\Sigma) : \|\mu\| \leq 2k\}$ is weak*-compact. Therefore, to complete the proof it remains to show that $core(\nu)$

[3] The weak*-topology and its properties can be found in, e.g., [1], [20] and [52].

is weak*-closed. Let $\{\mu_\alpha\}_\alpha$ be a net in $core(\nu)$ that weak*-converges to $\mu \in ba(\Sigma)$. Using the properties of the weak* topology, it is easy to see that $\mu \in core(\nu)$. Hence, $core(\nu)$ is weak*-closed. ■

Remark. When Σ is a σ-algebra, the condition of boundedness of the game in Proposition 3 is superfluous by the Nikodym Uniform Boundedness Theorem (see, e.g., [5, pp. 204-205]).

The core suggests some further taxonomy on games. A game ν is

11. *balanced* if its core is nonempty;
12. *totally balanced* if all its subgames ν_A have nonempty cores.[4]

We already observed that for a charge μ it holds $\mu = \overline{\mu}$. This property actually characterizes charges among balanced games.

Proposition 4 *A balanced game ν is a charge if and only if $\nu = \overline{\nu}$.*

Proof. The "only if" part is trivial. As to the "if part", let $\mu \in core(\nu)$. As $\nu \leq \mu \leq \overline{\nu}$, we have $\nu = \mu = \overline{\nu}$, as desired. ■

The next result characterizes balanced games directly in terms of properties of the game ν. It was proved by Bondareva [7] and Shapley [62] for finite games, and extended to infinite games by Schmeidler [55].

Theorem 5 *A bounded game is balanced if and only if, for all $\lambda_1, ..., \lambda_n \geq 0$ and all $A_1, ..., A_n \in \Sigma$, it holds*

$$\sum_{i=1}^{n} \lambda_i \nu(A_i) \leq \nu(\Omega) \quad \textit{whenever} \sum_{i=1}^{n} \lambda_i 1_{A_i} = 1_\Omega. \tag{4}$$

Proof. As the converse is trivial, we only show that ν is balanced provided it satisfies (4). By (4), $\nu(A) + \nu(A^c) \leq \nu(\Omega)$ for all A, so that $\nu \leq \overline{\nu}$. Let $\mathcal{E}$ be the collection of all finite subalgebras Σ_0 of Σ; for each $\Sigma_0 \in \mathcal{E}$ set

$$c(\Sigma_0) = \left\{\gamma \in \mathbb{R}^\Sigma : \nu(A) \leq \gamma(A) \leq \overline{\nu}(A) \text{ for each } A \in \Sigma \text{ and } \gamma_{|\Sigma_0} \text{ is a charge}\right\},$$

[4] The subgame ν_A is the restriction of ν on the induced algebra $\Sigma_A = \Sigma \cap A$ given by $\nu_A(B) = \nu(B)$ for all $\Sigma \ni B \subseteq A$.

where $\mathbb{R}^{\Sigma}$ is the collection of all set functions on Σ, and $\gamma_{|\Sigma_0}$ is the restriction of γ on Σ_0.

The set $c(\Sigma_0)$ is nonempty. In fact, as Σ_0 is finite and the restriction $\nu_{|\Sigma_0}$ satisfies (4), by [7] and [62] there exists a charge γ_0 on Σ_0 satisfying $\nu(A) \leq \gamma_0(A) \leq \overline{\nu}(A)$ for each $A \in \Sigma_0$. If we set

$$\gamma(A) = \begin{cases} \gamma_0(A) & \text{if } A \in \Sigma_0 \\ \nu(A) & \text{otherwise} \end{cases}$$

we have $\gamma \in c(\Sigma_0)$, so that $c(\Sigma_0) \neq \varnothing$.

Set $a = \inf_{A\in\Sigma} \nu(A)$ and $b = \sup_{A\in\Sigma} \overline{\nu}(A)$. Both a and b belong to $\mathbb{R}$ since ν is bounded, and so by the Tychonoff Theorem (see, e.g., [1, p. 52]) the set $\prod_{B\in\Sigma}[a,b]$ is compact in the product topology of $\mathbb{R}^{\Sigma}$. Clearly, $c(\Sigma_0) \subseteq \prod_{B\in\Sigma}[a,b]$. We want to show that $c(\Sigma_0)$ is actually a closed subset of $\prod_{B\in\Sigma}[a,b]$. Let γ_t be a net in $c(\Sigma_0)$ such that $\gamma_t \to \gamma \in \mathbb{R}^{\Sigma}$ in the product topology, i.e., $\gamma_t(A) \to \gamma(A)$ for all $A \in \Sigma$. For each A and each t, we have $\nu(A) \leq \gamma_t(A) \leq \overline{\nu}(A)$; hence, $\nu(A) \leq \gamma(A) \leq \overline{\nu}(A)$. For each t and for all disjoint A and B in Σ_0, we have $\gamma_t(A \cup B) = \gamma_t(A) + \gamma_t(B)$; hence, $\gamma(A \cup B) = \gamma(A) + \gamma(B)$. We conclude that $\gamma \in c(\Sigma_0)$, and so $c(\Sigma_0)$ is a closed (and so compact) subset of $\prod_{B\in\Sigma}[a,b]$.

If $\Sigma_0 \subseteq \Sigma_0'$, then $c(\Sigma_0') \subseteq c(\Sigma_0)$. Hence, denoted by $\widetilde{\Sigma}_0 \in \mathcal{E}$ the algebra generated by a finite sequence $\left\{\Sigma_0^i\right\}_{i=1}^n \subseteq \mathcal{E}$, we have

$$\emptyset \neq c\left(\widetilde{\Sigma}_0\right) \subseteq \bigcap_{i=1}^n c\left(\Sigma_0^i\right).$$

In other words, the collection of compact sets $\{c(\Sigma_0)\}_{\Sigma_0\in\mathcal{E}}$ satisfies the finite intersection property. In turn, this implies $\bigcap_{\Sigma_0\in\mathcal{E}} c(\Sigma_0) \neq \varnothing$ (see, e.g., [1, p. 38]), which means that there exists a charge γ such that $\nu(A) \leq \gamma(A) \leq \overline{\nu}(A)$ for each $A \in \Sigma$. Since $\gamma \in \prod_{B\in\Sigma}[a,b]$, the charge γ is bounded and so it belongs to $ba(\Sigma)$. We conclude that *core* $(\nu) \neq \varnothing$, as desired. ■

Remark. As observed by Kannai [34, pp. 229-230], for positive games Theorem 5 also follows from a result of Fan [23] on systems of linear inequalities in normed spaces.

Since countable additivity is a most useful technical property, it is natural to wonder when it is the case that a nonempty core actually contains some measures. The next example of Kannai [34] shows that this might well not happen.

Example 6 Let $\Omega = \mathbb{N}$ and consider the game $\nu : 2^{\mathbb{N}} \to \mathbb{R}$ defined by

$$\nu(A) = \begin{cases} 0 & A^c \text{ is infinite} \\ 1 & \text{else} \end{cases}$$

Here $core(\nu) \neq \varnothing$. In fact, let ∇ be any ultrafilter containing the filter of all sets having finite complements. The two-valued charge $u_\nabla : 2^{\mathbb{N}} \to \mathbb{R}$ defined by

$$u_\nabla(A) = \begin{cases} 1 & A \in \nabla \\ 0 & \text{else} \end{cases}$$

belongs to $core(\nu)$. On the other hand, $core(\nu) \cap ca(\Sigma) = \varnothing$. For, suppose *per contra* that $\mu \in core(\nu) \cap ca(\Sigma)$. For each $n \in \mathbb{N}$ we have $\mu(n) = \mu(\mathbb{N}) - \mu(\mathbb{N} - \{n\}) = 0$. The countable additivity of μ then implies $\mu(\mathbb{N}) = \sum_n \mu(n) = 0$, which contradicts $\mu(\mathbb{N}) = \nu(\mathbb{N}) = 1$. ▲

For positive games it is trivially true that $core(\nu) \subseteq ca(\Sigma)$ provided ν is continuous at Ω. In fact, for each monotone sequence $A_n \uparrow \Omega$ it holds

$$\nu(\Omega) = \mu(\Omega) \geq \lim_n \mu(A_n) \geq \lim_n \nu(A_n) = \nu(\Omega),$$

for all $\mu \in core(\nu)$. Hence, $\mu(\Omega) = \lim_n \mu(A_n)$, which implies $\mu \in ca(\Sigma)$. For signed games we have a more interesting result, based on [2, p. 173].

Proposition 7 *Given a balanced game ν, it holds $core(\nu) \subseteq ca(\Sigma)$ provided ν is continuous at both Ω and $\varnothing$.*

Proof. Consider $A_n \uparrow \Omega$. Let $\mu \in core(\nu)$. We want to show that $\mu(\Omega) = \lim_n \mu(A_n)$. Since $\mu(A_n) \geq \nu(A_n)$ for each n, by the continuity of ν at Ω we have $\liminf_n \mu(A_n) \geq \liminf_n \nu(A_n) = \nu(\Omega)$. On the other hand, since $A_n^c \downarrow \varnothing$ and ν is continuous at $\varnothing$, we have:

$$\limsup_n \mu(A_n) = \mu(\Omega) - \liminf_n \mu(A_n^c) \leq \nu(\Omega) - \liminf_n \nu(A_n^c) = \nu(\Omega).$$

In sum,

$$\limsup_n \mu(A_n) \leq \nu(\Omega) \leq \liminf_n \mu(A_n),$$

and so $\mu(\Omega) = \lim_n \mu(A_n)$, as desired. ■

The next example shows that in general these continuity properties are only sufficient for the core being contained in $ca(\Sigma)$.

Example 8 Let λ be the Lebesgue measure on $[0,1]$ and let $f:[0,1]\to\mathbb{R}$ be given by

$$f(x)=\begin{cases} x & 0\le x\le \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}<x<1 \\ 1 & x=1 \end{cases}$$

Consider the game $\nu(A)=f(\lambda(A))$ for each A. Though this game is not continuous at Ω, we have $core(\nu)=\{\lambda\}\in ca(\Sigma)$. For, let $\mu\in core(\nu)$. We want to show that $\mu=\lambda$. Given any A, there is a partition $\{A_i\}_{i=1}^n$ of A such that $\lambda(A_i)\le 1/2$. Hence, $\mu(A)=\sum_{i=1}^n\mu(A_i)\ge\sum_{i=1}^n\lambda(A_i)=\lambda(A)$. Since A was arbitrary, this implies $\mu\ge\lambda$, and so $\mu=\lambda$. ▲

Intuitively, this example works because the connection between the form of the game $\nu=f(\lambda)$ and its core is a bit "loose." Formally, there are gaps between ν and the core's lower envelope $\min_{\mu\in core(\nu)}\mu(A)$. For example, if A is such that $\lambda(A)=3/4$, then $\nu(A)=1/2<3/4=\min_{\mu\in core(\nu)}\mu(A)$.

To fix this problem, Schmeidler [57] introduced the following class of games: a game ν is

13. *exact* if it is balanced and $\nu(A)=\min_{\mu\in core(\nu)}\mu(A)$ for each A.

In other words, a game is exact if for each A there is $\mu\in core(\nu)$ such that $\nu(A)=\mu(A)$. Exact games can thus be viewed as games in which there is a tight connection between the form of the game and its core.

Schmeidler [57] provided a characterization of exact games in terms of the game ν, related to (4). Moreover, he was able to prove that for exact games continuity becomes a necessary and sufficient condition for the core to be a subset of $ca(\Sigma)$. To see why this is the case, we need a remarkable property of weak*-compact subsets of $ca(\Sigma)$, due to Bartle, Dunford and Schwartz (see [38] and the references therein contained). The result requires Σ to be a σ-algebra, a natural domain for continuous set functions.

Lemma 9 *If Σ is a σ-algebra, then a subset of $ca(\Sigma)$ is weak*-compact if and only if it is weakly compact.*

Remark. As the proof shows, this lemma is a consequence of the Dini Theorem when $K\subseteq ca^+(\Sigma)$.

Proof. It is enough to prove that a weak*-compact subset of $ca(\Sigma)$ is weakly compact, the converse being trivial. Suppose $K\subseteq ca(\Sigma)$ is weak*-compact.

Since K is bounded and weakly closed, by [20, Thm. IV.9.1] the set K is sequentially weakly compact if and only if, given any $A_n \uparrow \Omega$, for each $\varepsilon > 0$ there is a positive integer $n(\varepsilon)$ such that $|\mu(\Omega) - \mu(A_n)| < \varepsilon$ for all $\mu \in K$ and all $n \geq n(\varepsilon)$. In other words, if and only if the measures in K are uniformly countably additive.

For convenience, we only consider the case $K \subseteq ca^+(\Sigma)$ (see, e.g., [38] for the general case). For each $n \geq 1$ consider the evaluation functions $\phi_n : ba(\Sigma) \to \mathbb{R}$ defined by

$$\phi_n(\mu) = \mu(A_n) \quad \text{for each } \mu \in ba(\Sigma).$$

Moreover, let $\phi : ba(\Sigma) \to \mathbb{R}$ be defined by $\phi(\mu) = \mu(\Omega)$ for each $\mu \in ba(\Sigma)$. Both the function ϕ and each function ϕ_n are weak*-continuous, and the sequence $\{\phi_n\}_{n\geq 1}$ is increasing on K. As K is weak*-compact and

$$\lim_n \phi_n(\mu) = \lim_n \mu(A_n) = \mu(\Omega) = \phi(\mu) \quad \text{for each } \mu \in K,$$

by the Dini Theorem (see, e.g., [1, p. 55]) ϕ_n converges uniformly to ϕ. In turn, this easily implies the desired uniform countable additivity of the measures in K, and so K is sequentially weakly compact. By the Eberlein-Smulian Theorem (see, e.g., [1, p. 256]), K is then weakly compact as well. ■

Using this lemma we can prove the following result, due to Schmeidler [57] for positive games. Here $|\mu|(A)$ denotes the total variation of μ at A (see, e.g., [1, p. 360])

Theorem 10 *Let $\nu : \Sigma \to \mathbb{R}$ be an exact game defined on a σ-algebra Σ. Then, the following conditions are equivalent:*

(i) ν is continuous at Ω and $\varnothing$.

(ii) ν is continuous at each A.

(iii) core (ν) is a weakly compact subset of $ca(\Sigma)$.

(iv) there exists $\lambda \in ca^+(\Sigma)$ such that, given any A, for all $\varepsilon > 0$ there exists $\delta > 0$ such

$$\lambda(A) < \delta \Longrightarrow |\mu|(A) < \varepsilon \quad \textit{for all } \mu \in \textit{core}(\nu). \tag{5}$$

Remark. Inspection of the proof shows that when ν is positive, in (i) we can just assume continuity at Ω, while in (iv) we can choose λ so that it belongs to $core\,(\nu)$.

Proof. (ii) trivially implies (i), which in turn implies $core\,(\nu) \subseteq ca\,(\Sigma)$ by Proposition 7. By Proposition 3, $core\,(\nu)$ is weak*-compact, and so, by Lemma 9, it is weakly compact as well. Assume (iii) holds. Since $core\,(\nu)$ is a weakly compact subset of $ca\,(\Sigma)$, by [20, Thm. IV.9.2] there is $\lambda \in ca^+\,(\Sigma)$ such that (iv) holds. If ν is positive, following [12, p. 226] replace $1/2^i$ by $1/m_n$ at the bottom of [20, p. 307] to get $\lambda \in core\,(\nu)$.

It remains to show that (iv) implies (ii). Assume (iv). Since λ is countably additive, (5) implies that each $\mu \in core\,(\nu)$ is countably additive as well, i.e., $core\,(\nu) \subseteq ca\,(\Sigma)$. By Lemma 9, $core\,(\nu)$ is weakly compact. We are now ready to show that ν is continuous at each A. *Per contra*, suppose there is some A at which ν is not continuous, i.e., there is a sequence, say $A_n \uparrow A$ (the argument for $A_n \downarrow A$ is similar), and some $\eta > 0$ such that $|\nu\,(A_n) - \nu\,(A)| \geq \eta$. As ν is exact, for each n there is $\mu_n \in core\,(\nu)$ such that $\nu\,(A_n) = \mu_n\,(A_n)$. By the Eberlein-Smulian Theorem (see, e.g., [1, p. 256]), $core\,(\nu)$ is sequentially weakly compact as well. Hence, there is a suitable subsequence $\{\mu_{n_k}\}_{n_k}$ of $\{\mu_n\}_n$ such that μ_{n_k} weakly converges to some $\widetilde{\mu} \in core\,(\nu)$. By [20, Thm. IV.9.5], this means that $\lim_k \mu_{n_k}\,(A) = \widetilde{\mu}\,(A)$ for each A.

Now, consider:

$$\nu\,(A_{n_k}) = \mu_{n_k}\,(A_{n_k}) = \mu_{n_k}\,(A) - \mu_{n_k}\,(A \setminus A_{n_k})\,. \tag{6}$$

Clearly, $A \setminus A_{n_k} \downarrow \varnothing$. Since $core\,(\nu)$ is weakly compact, by [20, Thm. IV.9.1] the measures in $core\,(\nu)$ are uniformly countably additive, and so for each $\varepsilon > 0$ there is $k\,(\varepsilon) \geq 1$ such that $|\mu\,(A \setminus A_{n_k})| < \varepsilon$ for all $\mu \in core\,(\nu)$ and all $k \geq k\,(\varepsilon)$. In particular, $\left|\mu_{n_k}\,(A \setminus A_{n_k})\right| < \varepsilon$ for all $k \geq k\,(\varepsilon)$. As ε is arbitrary, this implies $\lim_k \mu_{n_k}\,(A \setminus A_{n_k}) = 0$. By (6), we then have

$$\lim_k \nu\,(A_{n_k}) = \lim_k \mu_{n_k}\,(A_{n_k}) = \widetilde{\mu}\,(A) \geq \nu\,(A)\,. \tag{7}$$

On the other hand, there exists a $\widehat{\mu} \in core\,(\nu)$ such that $\widehat{\mu}\,(A) = \nu\,(A)$. Hence,

$$\nu\,(A) = \widehat{\mu}\,(A) = \lim_k \widehat{\mu}\,(A_{n_k}) \geq \lim_k \nu\,(A_{n_k})\,. \tag{8}$$

Putting together (7) and (8), we get $\nu\,(A) = \lim_{n_k} \nu\,(A_{n_k})$, thus contradicting $|\nu\,(A_n) - \nu\,(A)| \geq \eta$. We conclude that ν is continuous at A, as desired. ■

Point (iv) is noteworthy. It says that the continuity of ν guarantees the existence of a positive control measure λ for *core* (ν), that is, a measure $\lambda \in ca^+(\Sigma)$ such that $\mu << \lambda$ for all $\mu \in core(\nu)$.

This is a very useful property; *inter alia*, it implies that *core* (ν) can be identified with a subset of $L_1(\lambda)$, the set of all (equivalence classes) of Σ-measurable functions that are integrable with respect to λ. In fact, by the Radon-Nikodym Theorem (see, e.g., [1, p. 437]) to each $\mu << \lambda$ corresponds a unique $f \in L_1(\lambda)$ such that $\mu(A) = \int_A f d\lambda$ for all A.

Summing up:

Corollary 11 *Let $\nu : \Sigma \to \mathbb{R}$ be an exact game defined on a σ-algebra Σ. Then, ν is continuous at Ω and $\varnothing$ if and only if there is $\lambda \in ca^+(\Sigma)$ such that core (ν) is a weakly compact subset of $L_1(\lambda)$.*

Proof. Set $ca(\lambda) = \{\mu \in ca : \mu << \lambda\}$. By the Radon-Nikodym Theorem, there is an isometric isomorphism between $ca(\lambda)$ and $L_1(\lambda)$ determined by the formula $\mu(A) = \int_A f d\lambda$ (see [20, p. 306]). Hence, a subset is weakly compact in $ca(\lambda)$ if and only if it is in $L_1(\lambda)$ as well. ■

It is sometimes useful to know when the core of a continuous game consists of non-atomic measures. We close the section by studying this problem, which also provides a further illustration of the usefulness of the control measure λ.

In order to do so, we need to introduce null sets. Given a game ν, a set N is *ν-null* if

$$\nu(N \cup A) = \nu(A) \qquad \text{for all } A \in \Sigma. \tag{9}$$

The next lemma collects some basic properties of null sets.

Lemma 12 *Given a game ν, let N be a ν-null set. Then:*

(i) each subset $B \subseteq N$ is ν-null;

(ii) $\nu(B) = 0$ and $\nu(A \setminus B) = \nu(A)$ for any $B \subseteq N$;

(iii) N is $\overline{\nu}$-null.

Proof. (i). Let $B \subseteq N$ and let A be any set in Σ. By (9),

$$\nu(B \cup A) = \nu(B \cup A \cup N) = \nu(A \cup N) = \nu(A),$$

and so B is ν-null.

(ii) If we put $A = \varnothing$ in (9), we get $\nu(N) = 0$. By (i), each $B \subseteq N$ is ν-null, so that $\nu(B) = 0$ by what we have just established. It remains to show that $\nu(A \setminus B) = \nu(A)$ for any $B \subseteq N$. By (i), $A \cap B$ is ν-null. Hence,

$$\nu(A \setminus B) = \nu((A \setminus B) \cup (A \cap B)) = \nu(A),$$

as desired.

(iii) Let A be any set in Σ. By (ii) we then have:

$$\overline{\nu}(A \cup N) = \nu(\Omega) - \nu(A^c \setminus N) = \nu(\Omega) - \nu(A^c) = \overline{\nu}(A),$$

as desired. ■

For a charge μ, a set N is μ-null if and only if $|\mu|(N) = 0$. For, suppose N is μ-null. We have (see, e.g., [1, p. 360]):

$$|\mu|(N) = \sup\{|\mu(B)| + |\mu(N \setminus B)| : B \subseteq N\},$$

and so point (ii) of Lemma 12 implies $|\mu|(N) = 0$. Conversely, suppose $|\mu|(N) = 0$. Then, $|\mu(B)| = 0$ for each $B \subseteq N$, and so

$$\mu(A \cup N) = \mu(A \cup N \setminus A) = \mu(A) + \mu(N \setminus A) = \mu(A)$$

for each set $A \in \Sigma$. We conclude that N is μ-null, as desired.

Given two games ν_1 and ν_2, we say that ν_1 is *absolutely continuous with respect to* ν_2 (written $\nu_1 \ll \nu_2$) when each ν_2-null set is ν_1-null; we say that the two games are *equivalent* (written $\nu_1 \equiv \nu_2$) when a set is ν_1-null if and only if it is ν_2-null. In the special case of charges we get back to the standard definitions of absolute continuity (see, e.g., [1, p. 363]).

Given a balanced game ν, we have $\mu \ll \nu$ for each $\mu \in core(\nu)$. For, let $m \in core(\nu)$ and suppose N is ν-null. For each $A \subseteq N$, we have $m(A) \geq \nu(A) = 0$, and $m(A^c) \geq \nu(A^c) = \nu(\Omega) = m(\Omega) = m(A) + m(A^c)$. Hence, $m(A) = 0$ for all $A \subseteq N$, namely, $|m|(N) = 0$. For continuous exact games we have the following deeper result, due to Schmeidler [57, Thm 3.10], which provides a further useful property of the control measure λ.

Lemma 13 *Given an exact and continuous game ν defined on a σ-algebra Σ, let λ be the control measure of Theorem 10. Then, $\nu \equiv \lambda$.*

Proof. By [20, Thm. IV.9.2], we have

$$\lambda = \sum_{n=1}^{\infty} \frac{2^{-n}}{k_n} \sum_{i=1}^{k_n} |\mu_i^n| \tag{10}$$

with each $\mu_i^n \in core\,(\nu)$. Let N be ν-null. As $\mu << \nu$ for each $\mu \in core\,(\nu)$, N is μ-null for each $\mu \in core\,(\nu)$. Hence, $|\mu|\,(N) = 0$ for all $\mu \in core\,(\nu)$. By (10), $\lambda\,(N) = 0$. Therefore N is λ-null.

Conversely, suppose $\lambda\,(N) = 0$. As $\mu << \lambda$ for each $\mu \in core\,(\nu)$, we have $|\mu|\,(N) = 0$ for each $\mu \in core\,(\nu)$. By exactness, there are $\mu, \mu' \in core\,(\nu)$ such that:

$$\nu\,(N \cup F) = \mu\,(N \cup F) = \mu\,(F) \geq \nu\,(F) = \mu'\,(F) = \mu'\,(N \cup F) \geq \nu\,(N \cup F)$$

and so N is ν-null. We conclude that $\nu \equiv \lambda$, as desired. ■

A game ν is *non-atomic* if for each ν-nonnull set A there is a set $B \subseteq A$ such that both B and $A \backslash B$ are ν-nonnull. In particular, a charge μ is non-atomic if and only if for each $|\mu|\,(A) > 0$ there is $B \subseteq A$ such that $0 < |\mu|\,(B) < |\mu|\,(A)$. In turn, this is equivalent to require that for each $\mu\,(A) \neq 0$ there is $B \subseteq A$ such that both $\mu\,(B) \neq 0$ and $\mu\,(A \backslash B) \neq 0$ (see [5, pp. 141-142]).

We can now state and prove the announced result on “non-atomic” cores.

Proposition 14 *Let ν be a continuous exact game defined on σ-algebra Σ. Then, ν is non-atomic if and only if core (ν) consists of non-atomic measures.*

Proof. “If” part. Suppose ν is non-atomic. By Lemma 13, λ as well is non-atomic. In turn, this implies that each $\mu \in core\,(\nu)$ is non-atomic. In fact, let $|\mu|\,(A) \neq 0$ for some A, so that $\lambda\,(A) > 0$. Since λ is non-atomic, there is a partition $A_{1/2}^1, B_{1/2}^1$ of A such that $\lambda\left(A_{1/2}^1\right) = \lambda\left(B_{1/2}^1\right) = \frac{1}{2}\lambda(A)$ (see [5, Thm 5.1.6]). If $0 < |\mu|\left(A_{1/2}^1\right) < |\mu|\,(A)$ or $0 < |\mu|\left(B_{1/2}^1\right) < |\mu|\,(A)$, we are done. Suppose, in contrast, that either $|\mu|\left(A_{1/2}^1\right) = |\mu|\,(A)$ or $|\mu|\left(B_{1/2}^1\right) = |\mu|\,(A)$. Without loss, let $|\mu|\left(A_{1/2}^1\right) = |\mu|\,(A)$. Let $A_{1/2}^2$ and $B_{1/2}^2$ be a partition of $A_{1/2}^1$ such that $\lambda\left(A_{1/2}^2\right) = \lambda\left(B_{1/2}^2\right) = \frac{1}{2}\lambda(A_{1/2}^1)$. If $0 < |\mu|\left(A_{1/2}^2\right) < |\mu|\,(A_{1/2}^1)$ or $0 < |\mu|\left(B_{1/2}^2\right) < |\mu|\,(A_{1/2}^1)$, we are done.

Suppose, in contrast, that either $|\mu|\left(A^2_{1/2}\right) = |\mu|(A^1_{1/2})$ or $|\mu|\left(B^2_{1/2}\right) = |\mu|(A^1_{1/2})$. Without loss, let $|\mu|\left(A^2_{1/2}\right) = |\mu|(A^1_{1/2})$. By proceeding in this way, either we find a set $B \subseteq A$ such that $0 < |\mu|(B) < |\mu|(A)$ or we can construct a chain $\left\{A^n_{1/2}\right\}_{n\geq 1}$ such that $\lambda\left(A^n_{1/2}\right) = \frac{1}{2^n}\lambda(A)$ and $|\mu|\left(A^n_{1/2}\right) = |\mu|(A)$ for all $n \geq 1$. Hence, being $\bigcap_{n\geq 1} A^n_{1/2} \in \Sigma$, $\lambda\left(\bigcap_{n\geq 1} A^n_{1/2}\right) = 0$ and $|\mu|\left(\bigcap_{n\geq 1} A^n_{1/2}\right) = |\mu|(A) > 0$. Since $\mu \ll \lambda$, this is impossible, and so there exists some set $\Sigma \ni B \subseteq A$ such that $0 < |\mu|(B) < |\mu|(A)$. We conclude that μ is non-atomic, as desired.

"Only if." Suppose that each $\mu \in core(\nu)$ is non-atomic. Set $\lambda_n = (2^{-n}/k_n)\sum_{i=1}^{k_n} |\mu^n_i|$ in (10). Then, $\lambda = \sum_{n=1}^{\infty} \lambda_n$. Each positive measure λ_n is non-atomic. For, suppose $\lambda_n(A) > 0$. There is some $|\mu^n_i|$ such that $|\mu^n_i|(A) > 0$. Hence, there is $B \subseteq A$ such that $|\mu^n_i|(B) > 0$ and $|\mu^n_i|(A\backslash B) > 0$. Since $\lambda_n \geq |\mu^n_i|$, we then have $\lambda_n(B) > 0$ and $\lambda_n(A\backslash B) > 0$, as desired. Since each λ_n is non-atomic, λ as well is non-atomic. For, suppose $\lambda(A) > 0$. There is some λ_n such that $\lambda_n(A) > 0$. Hence, there is $B \subseteq A$ such that $\lambda_n(B) > 0$ and $\lambda_n(A\backslash B) > 0$. Since $\lambda \geq \lambda_n$, we then have $\lambda(B) > 0$ and $\lambda(A\backslash B) > 0$, and so λ is non-atomic. By Lemma 13, $\nu \equiv \lambda$. As λ is non-atomic, this implies that ν as well is non-atomic, as desired. ■

3 Choquet Integrals

Given a game $\nu : \Sigma \to \mathbb{R}$ and a real-valued function $f : \Omega \to \mathbb{R}$, a natural question is whether there is a meaningful way to write an integral $\int f d\nu$ that extends the standard notions of integrals for additive games.

Fortunately, Choquet [11, p. 265] has shown that it is possible to develop a rich theory of integration in a non-additive setting. As usual with notions of integration, we will present Choquet's integral in a few steps, beginning with positive functions.

3.1 Positive Functions

A function $f : \Omega \to \mathbb{R}$ is Σ-*measurable* if $f^{-1}(I) \in \Sigma$ for each open and each closed interval I of $\mathbb{R}$ (see [20, p. 240]). The set of all bounded Σ-measurable $f : \Omega \to \mathbb{R}$ is denoted by $B(\Sigma)$.

Proposition 15 *The set $B(\Sigma)$ is a lattice. If, in addition, Σ is a σ-algebra, then $B(\Sigma)$ is a vector lattice.*

Proof. Let $f, g \in B(\Sigma)$. We only prove that $(f \vee g)^{-1}(a, b) \in \Sigma$ for any open (possibly unbounded) interval $(a, b) \subseteq \mathbb{R}$, the other cases being similar. For each $t \in \mathbb{R}$, the following holds:

$$\begin{aligned} (f \vee g > t) &= (f > t) \cup (g > t) \\ (f \vee g < t) &= (f < t) \cap (g < t). \end{aligned}$$

Hence,

$$\begin{aligned} (f \vee g)^{-1}(a, b) &= (f \vee g > a) \cap (f \vee g < b) \\ &= ((f > a) \cup (g > a)) \cap ((f < b) \cap (g < b)) \in \Sigma, \end{aligned}$$

as desired. Finally, the fact that $B(\Sigma)$ is a vector space when Σ is a σ-algebra is a standard result in measure theory (see [1, Thm 4.26]). ■

Given a capacity $\nu : \Sigma \to \mathbb{R}$ and a positive Σ-*measurable* function $f : \Omega \to \mathbb{R}$, the *Choquet integral* of f with respect to ν is given by:

$$\int f d\nu = \int_0^\infty \nu(\{\omega \in \Omega : f(\omega) \geq t\}) \, dt, \tag{11}$$

where on the right we have a Riemann integral. To see why the Riemann integral is well defined, first observe that

$$f^{-1}([t, +\infty)) = \{\omega \in \Omega : f(\omega) \geq t\} \in \Sigma \quad \text{for each } t \in \mathbb{R}.$$

Set $E_t = \{\omega \in \Omega : f(\omega) \geq t\}$; the *survival function* $G_\nu : \mathbb{R} \to \mathbb{R}$ of f with respect to ν is defined by $G_\nu(t) = \nu(E_t)$ for each $t \in \mathbb{R}$. Using this function, we can write (11) as $\int f d\nu = \int_0^\infty G_\nu(t) \, dt$. The family $\{E_t\}_{t \in \mathbb{R}}$ is a chain, with $E_t \supseteq E_{t'}$ if $t \leq t'$.[5] Since ν is a capacity, we have $\nu(E_t) \geq \nu(E_{t'})$ if $t \leq t'$, and so G_ν is a decreasing function. Moreover, since f is both positive and bounded, the function G_ν is positive, decreasing and with compact support. By standard results on Riemann integration, we conclude that the Riemann integral $\int_0^{+\infty} G_\nu(t) \, dt$ exists, and so the Choquet integral (11) is well defined.

[5] A collection $\mathcal{C}$ in Σ is *chain* if for each A and B in $\mathcal{C}$ it holds either $A \subseteq B$ or $B \subseteq A$. Throughout we assume that $\varnothing, \Omega \in \mathcal{C}$.

The Choquet integral $\int f d\nu$ reduces to the standard additive integral when ν is additive. Given a positive charge μ and a function f in $B(\Sigma)$, let $\widetilde{\int} f d\mu$ be the standard additive integral for charges (see, e.g., [1, p. 399] and [5, pp. 115-121]).

Proposition 16 *Given a positive function $f \in B(\Sigma)$ and a positive charge $\mu \in ba(\Sigma)$, it holds*

$$\widetilde{\int} f d\mu = \int \mu(f \geq t)\, dt = \int f d\mu.$$

Proof. We use an argument of [53, p. 172]. Set $E_t = (f \geq t)$. Given $\omega \in \Omega$, we have

$$\int_0^\infty 1_{E_t}(\omega)\, dt = \int_0^\infty 1_{[0,f(\omega)]}(t)\, dt = \int_0^{f(\omega)} dt = f(\omega).$$

Equivalently, $f(\omega) = \int_0^\infty 1_{E_t}(\omega)\, d\lambda$, where λ is the Lebesgue measure on $\mathbb{R}$. By the Fubini Theorem for the integral $\widetilde{\int}$ (see, e.g., [41]), we can write

$$\begin{aligned} \widetilde{\int} f d\mu &= \widetilde{\int}_\Omega \left(\int_0^\infty 1_{E_t}(\omega)\, d\lambda \right) d\mu = \int_0^\infty \left(\widetilde{\int}_\Omega 1_{E_t}(\omega)\, d\mu \right) d\lambda \\ &= \int_0^\infty \mu(f \geq t)\, d\lambda = \int_0^\infty \mu(f \geq t)\, dt, \end{aligned}$$

as desired. ■

We close by observing that in defining Choquet integrals we could have equivalently used the "strict" upper sets $(f > t)$.

Proposition 17 *Let ν be a capacity and f a positive function in $B(\Sigma)$. Then,*

$$\int_0^\infty \nu(f \geq t)\, dt = \int_0^\infty \nu(f > t)\, dt.$$

Proof. As before, set $G_\nu(t) = \nu(f \geq t)$ for each $t \in \mathbb{R}$. Moreover, set $G'_\nu(t) = \nu(f > t)$ for each $t \in \mathbb{R}$. We have $(f \geq t + 1/n) \subseteq (f > t) \subseteq (f \geq t)$ for each $t \in \mathbb{R}$, and so $G_\nu(t + 1/n) \leq G'_\nu(t) \leq G_\nu(t)$ for each $t \in \mathbb{R}$. If G_ν

is continuous at t, we have $G_\nu(t) = \lim_n G_\nu(t+1/n) \le G'_\nu(t) \le G_\nu(t)$, so that $G'_\nu(t) = G_\nu(t)$.

On the other hand, as G_ν is a decreasing function, it is continuous except on an at most countable set $T \subseteq \mathbb{R}$. As a result, it holds $G'_\nu(t) = G_\nu(t)$ for all $t \notin T$, which in turn implies $\int_0^\infty G'_\nu(t)\,dt = \int_0^\infty G_\nu(t)\,dt$ by standard results on Riemann integration. ■

3.2 General Functions

We now extend the definition of the Choquet integral to general Σ-measurable functions. In the previous subsection we have defined the Choquet integral on $B^+(\Sigma)$, the cone of all positive elements of $B(\Sigma)$. Each capacity ν induces a functional $\nu_c : B^+(\Sigma) \to \mathbb{R}$ on this cone, given by $\nu_c(f) = \int f d\nu$ for each $f \in B^+(\Sigma)$. If f is a characteristic function 1_A, we get $\nu_c(1_A) = \int 1_A d\nu = \nu(A)$; thus, the functional ν_c – which we call the *Choquet functional* – can be viewed as an extension of the capacity ν from Σ to $B^+(\Sigma)$.

Our problem of defining a Choquet integral on $B(\Sigma)$ can be viewed as the problem of how to extend the Choquet functional on the entire space $B(\Sigma)$. In principle, there are many different ways to extend it. To make the extension problem meaningful we have to set a *desideratum* for the extension, that is, a property we want it to satisfy.

A natural property to require is that the extended functional $\nu_c : B(\Sigma) \to \mathbb{R}$ be translation invariant, that is, $\nu_c(f + \alpha 1_\Omega) = \nu_c(f) + \alpha\nu_c(1_\Omega)$ for each $\alpha \in \mathbb{R}$ and each $f \in B(\Sigma)$. The next result shows that this *desideratum* pins down the extension to a particular form.

Proposition 18 *A Choquet functional $\nu_c : B^+(\Sigma) \to \mathbb{R}$ induced by a capacity admits a unique translation invariant extension, given by*

$$\int_0^\infty \nu(f \ge t)\,dt + \int_{-\infty}^0 [\nu(f \ge t) - \nu(\Omega)]\,dt \qquad (12)$$

for each $f \in B(\Sigma)$, where on the right we have two Riemann integrals.

Proof. Set

$$\widehat{\nu}_c(f) = \int_0^\infty \nu(f \ge t)\,dt + \int_{-\infty}^0 [\nu(f \ge t) - \nu(\Omega)]\,dt.$$

The functional $\widehat{\nu}_c$ is well defined and some simple algebra shows that it is translation invariant and that it reduces to the Choquet integral when $f \in B^+(\Sigma)$. Assume $\widetilde{\nu} : B(\Sigma) \to \mathbb{R}$ is a translation invariant functional such that $\widetilde{\nu}(f) = \nu_c(f)$ whenever $f \in B^+(\Sigma)$. We want to show that $\widetilde{\nu}$ satisfies (12), so that $\widetilde{\nu} = \widehat{\nu}_c$.

Let $f \in B(\Sigma)$ be such that $\inf f = \gamma < 0$. By translation invariance, $\widetilde{\nu}(f-\gamma) = \widetilde{\nu}(f) - \gamma\widetilde{\nu}(1_\Omega)$. As $f-\gamma$ belongs to $B^+(\Sigma)$, we can then write:

$$\begin{aligned}
\widetilde{\nu}(f) &= \widetilde{\nu}(f-\gamma) + \gamma\widetilde{\nu}(1_\Omega) \\
&= \nu_c(f-\gamma) + \gamma\nu_c(1_\Omega) \\
&= \int_0^\infty \nu((f-\gamma) \geq t)\, dt + \gamma\nu_c(1_\Omega) \\
&= \int_0^\infty \nu(f \geq t+\gamma)\, dt + \gamma\nu_c(1_\Omega) \\
&= \int_\gamma^\infty \nu(f \geq \tau)\, d\tau + \gamma\nu_c(1_\Omega) \\
&= \int_\gamma^0 \nu(f \geq \tau)\, d\tau + \int_0^\infty \nu(f \geq \tau)\, d\tau - \int_\gamma^0 \nu(\Omega)\, d\tau
\end{aligned}$$

where the penultimate equality is due to the change of variable $\tau = t + \gamma$.

As $[\nu(f \geq \tau) - \nu(\Omega)] = 0$ for all $\tau \leq \gamma$, the following holds:

$$\widetilde{\nu}(f) = \int_0^\infty \nu(f \geq \tau)\, d\tau + \int_{-\infty}^0 (\nu(f \geq \tau) - \nu(\Omega))\, d\tau.$$

Hence, $\widetilde{\nu} = \widehat{\nu}_c$, as desired. ■

Before moving on, observe that the Riemann integrals in (12) exist even if ν is a game of bounded variation, that is, if $\nu \in bv(\Sigma)$. In fact, for each such game there exist two capacities ν_1 and ν_2 with $\nu = \nu_1 - \nu_2$. Hence, $\nu(f \geq t) = \nu_1(f \geq t) - \nu_2(f \geq t)$ for each $t \in \mathbb{R}$, and so $\nu(f \geq t)$ is a function of bounded variation in t. The Riemann integrals in (12) then exist by standard results on Riemann integrals.

In view of Proposition 18 and the above observation, next we define the Choquet integral for functions in $B(\Sigma)$ with respect to games in $bv(\Sigma)$ as the translation invariant extension of the definition given in (11) for positive functions.

Definition 19 *Given a game $\nu \in bv(\Sigma)$ and a function $f \in B(\Sigma)$, the Choquet integral $\int f d\nu$ is defined by*

$$\int f d\nu = \int_0^\infty \nu(f \geq t)\, dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(\Omega)]\, dt. \tag{13}$$

The associated Choquet functional $\nu_c : B(\Sigma) \to \mathbb{R}$ is given by $\nu_c(f) = \int f d\nu$ for each $f \in B(\Sigma)$.

Translation invariance and Proposition 16 imply that when ν is a bounded charge, the Choquet integral $\int f d\nu$ of a $f \in B(\Sigma)$ reduces to the standard additive integral. Moreover, it is easy to check that Proposition 17 holds for general Choquet integrals, that is,

$$\int f d\nu = \int_0^\infty \nu(f > t)\, dt + \int_{-\infty}^0 [\nu(f > t) - \nu(\Omega)]\, dt.$$

Finally, the Choquet integral (13) is well defined for all finite games since they belong to $bv(\Sigma)$. As in the finite case $B(\Sigma) = \mathbb{R}^\Omega$, this means that finite games induce Choquet functionals $\nu_c : \mathbb{R}^\Omega \to \mathbb{R}$.

Example 20 Given a nonempty coalition A, the *unanimity game* $u_A : \Sigma \to \mathbb{R}$ is the two-valued convex game defined by

$$u_A(B) = \begin{cases} 1 & A \subseteq B \\ 0 & \text{else} \end{cases}$$

for all $B \in \Sigma$. For each $f \in B(\Sigma)$ it holds $\int f du_A = \inf_{\omega \in A} f(\omega)$. In fact, we have $A \subseteq (f \geq t)$ if and only if $t \leq \inf_{\omega \in A} f(\omega)$, and so $G_{u_A}(t) = 1_{(-\infty, \inf_{\omega \in A} f(\omega)]}(t)$. ▲

Example 21 Let $\Omega = \{\omega_1, \omega_2\}$ and suppose ν is a capacity on 2^Ω with $0 < \nu(\omega_1) < 1$, $0 < \nu(\omega_2) < 1$, and $\nu(\Omega) = 1$. Then, $\nu_c : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\nu_c(x_1, x_2) = \begin{cases} x_1(1 - \nu(\omega_2)) + x_2 \nu(\omega_2) & \text{if } x_2 \geq x_1, \\ x_1 \nu(\omega_1) + x_2(1 - \nu(\omega_1)) & \text{if } x_2 < x_1 \end{cases}$$

Given any $k \in \mathbb{R}$, the level curve $\{(x_1, x_2) \in \mathbb{R}^2 : \nu_c(x_1, x_2) = k\}$ is

$$\begin{cases} x_2 = \frac{k}{\nu(\omega_2)} - \frac{1-\nu(\omega_2)}{\nu(\omega_2)} x_1 & \text{if } x_2 \geq x_1, \\ x_2 = \frac{k}{1-\nu(\omega_1)} - \frac{\nu(\omega_1)}{1-\nu(\omega_1)} x_1 & \text{if } x_2 < x_1 \end{cases}$$

As a result, the level curve is a straight line when ν is a charge – i.e., when $\nu(\omega_1) + \nu(\omega_2) = 1$ – and it has, in contrast, a kink at the 45 degree line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$ when ν is not a charge. The non additivity of ν is thus reflected by kinks in the level curves. In general, level curves of Choquet integrals are not affine spaces, unless the game is a charge. ▲

A function f in $B(\Sigma)$ is *simple* if it is finite-valued, that is, if the set $\{f(\omega) : \omega \in \Omega\}$ is finite. Each simple function f admits a unique representation $f = \sum_{i=1}^k \alpha_i 1_{A_i}$, where $\{A_i\}_{i=1}^k \subseteq \Sigma$ is a suitable partition of Ω and $\alpha_1 > \cdots > \alpha_k$. Using this representation, we can rewrite formula (13) in a couple of equivalent ways, which are sometimes useful (see, for example, the discussion of the Choquet Expected Utility model of Schmeidler [58] in chapter 1).

Proposition 22 *Given a game $\nu \in bv(\Sigma)$ and a simple function $f \in B(\Sigma)$, it holds*

$$\int f d\nu = \sum_{i=1}^k (\alpha_i - \alpha_{i+1}) \nu\left(\bigcup_{j=1}^i A_j\right) = \sum_{i=1}^k \alpha_i \left(\nu\left(\bigcup_{j=0}^i A_j\right) - \nu\left(\bigcup_{j=0}^{i-1} A_j\right)\right),$$

where we set $\alpha_{k+1} = 0$ and $A_0 = \varnothing$.

Proof. It is enough to prove the first equality, the other being a simple rearrangement of its terms. Let f be positive, so that $\alpha_k \geq 0$. If $t > \alpha_1$, then $\{\omega \in \Omega : f(\omega) \geq t\} = \emptyset$. If $t \in (\alpha_{i+1}, \alpha_i]$, then (recall that $\alpha_{k+1} = 0$):

$$\{\omega \in \Omega : f(\omega) \geq t\} = \bigcup_{j=1}^i A_j.$$

Hence,

$$\nu(f \geq t) = \sum_{i=1}^k \nu\left(\bigcup_{j=1}^i A_j\right) 1_{(\alpha_{i+1}, \alpha_i]}(t) \quad \text{for each } t \in \mathbb{R}^+,$$

so that

$$\begin{aligned}\int f d\nu &= \int_0^\infty \nu(f \geq t)\, dt = \int_0^\infty \sum_{i=1}^k \nu\left(\bigcup_{j=1}^i A_j\right) 1_{(\alpha_{i+1}, \alpha_i]}(t)\, dt \\ &= \sum_{i=1}^k \nu\left(\bigcup_{j=1}^i A_j\right) \int_0^\infty 1_{(\alpha_{i+1}, \alpha_i]}(t)\, dt = \sum_{i=1}^k (\alpha_i - \alpha_{i+1}) \nu\left(\bigcup_{j=1}^i A_j\right),\end{aligned}$$

as desired. This proves the first equality for a positive f. The case of a general f is easily obtained using translation invariance. ■

When $\nu \in ba(\Sigma)$, the above formulae reduce to

$$\int f d\nu = \sum_{i=1}^{k} \alpha_i \nu(A_i),$$

which is the standard integral of f with respect to the charge ν.

Example 23 Let $P : \Sigma \to [0,1]$ be a probability charge with range $R(P) = \{P(A) : A \in \Sigma\}$. Given a real-valued function $g : R(P) \to \mathbb{R}$, the game $\nu = f(P)$ is called a *scalar measure game*. It holds

$$\int f d\nu = \int_0^\infty g(P(f \geq t))\, dt + \int_{-\infty}^0 [g(P(f \geq t)) - g(1)]\, dt.$$

The right hand side becomes

$$\sum_{i=1}^{k} \alpha_i \left[g\left(P\left(\bigcup_{j=0}^{i} A_j\right)\right) - g\left(P\left(\bigcup_{j=0}^{i-1} A_j\right)\right)\right]$$

when f is a simple function. This is a familiar formula in Rank Dependent Expected Utility (see chapters 1 and 2). ▲

3.3 Basic Properties

We begin by collecting a few basic properties of Choquet integrals. Here, $\|\cdot\|$ on $bv(\Sigma)$ is the variation norm given by (3), while $\geq$ and $\|\cdot\|$ on $B(\Sigma)$ are the pointwise order and the supnorm, respectively.[6]

Proposition 24 *Suppose $\nu_c : B(\Sigma) \to \mathbb{R}$ is the Choquet functional induced by a game $\nu \in bv(\Sigma)$. Then:*

(i) (Positive homogeneity): $\nu_c(\alpha f) = \alpha \nu_c(f)$ for each $\alpha \geq 0$.

(ii) (Translation invariance): $\nu_c(f + \alpha 1_\Omega) = \nu_c(f) + \alpha \nu_c(1_\Omega)$ for each $\alpha \in \mathbb{R}$.

[6] That is, $f \geq g$ if $f(\omega) \geq g(\omega)$ for each $\omega \in \Omega$, and $\|f\| = \sup_{\omega \in \Omega} |f(\omega)|$.

(iii) (Monotonicity): $\nu_c(f) \geq \nu_c(g)$ if $f \geq g$, provided ν is a capacity.

(iv) (Lipschitz continuity): for all $f, g \in B(\Sigma)$,

$$|\nu_c(f) - \nu_c(g)| \leq \|\nu\| \|f - g\|. \tag{14}$$

Proof. Properties (i) and (ii) are easily established. To see that (iii) holds it is enough to observe that, being ν a capacity, it holds $\nu(g \geq t) \leq \nu(f \geq t)$ for each $t \in \mathbb{R}$ since $f \geq g$ implies $(g \geq t) \subseteq (f \geq t)$ for each $t \in \mathbb{R}$.

As to (iv), suppose first that ν is a capacity. Assume $\nu_c(f) \geq \nu_c(g)$ (the other case is similar). As $f \leq g + \|f - g\|$, by (ii) and (iii) we have $\nu_c(f) \leq \nu_c(g) + \|f - g\| \nu(\Omega)$. This implies

$$|\nu_c(f) - \nu_c(g)| \leq \nu(\Omega) \|f - g\|, \tag{15}$$

which is (14) when ν is monotonic. For, in this case $\|\nu\| = \nu(\Omega)$.

Now, let $\nu \in bv(\Sigma)$. By [2, p. 28], ν can be written as $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are capacities such that $\|\nu\| = \nu^+(\Omega) + \nu^-(\Omega)$. By (15), we then have:

$$|\nu_c(f) - \nu_c(g)| \leq \left[\nu^+(\Omega) + \nu^-(\Omega)\right] \|f - g\|,$$

as desired. ■

If a game ν belongs to $bv(\Sigma)$, its dual $\overline{\nu}$ as well belongs to $bv(\Sigma)$. The Choquet functional $\overline{\nu}_c$ is therefore well defined and next we show that it can be viewed as the dual functional of ν_c.

Proposition 25 *Let $\nu \in bv(\Sigma)$. Then,*

$$\overline{\nu}_c(f) = -\nu_c(-f)$$

for each $f \in B(\Sigma)$. If, in addition, ν is balanced, then

$$\nu_c(f) \leq \mu(f) \leq \overline{\nu}_c(f)$$

for each $f \in B(\Sigma)$ and each $\mu \in core(\nu)$.

Proof. Given $f \in B(\Sigma)$, we have:

$$\begin{aligned}
\overline{\nu}_c(f) &= \int_0^\infty \overline{\nu}(f \geq t)\, dt + \int_{-\infty}^0 [\overline{\nu}(f \geq t) - \overline{\nu}(\Omega)]\, dt \\
&= \int_0^\infty [\nu(\Omega) - \nu(f < t)]\, dt + \int_{-\infty}^0 -\nu(f < t)\, dt \\
&= \int_0^\infty [\nu(\Omega) - \nu(f \leq t)]\, dt - \int_{-\infty}^0 \nu(f \leq t)\, dt \\
&= \int_0^\infty [\nu(\Omega) - \nu(-f \geq -t)]\, dt - \int_{-\infty}^0 \nu(-f \geq -t)\, dt \\
&= \int_{-\infty}^0 [\nu(\Omega) - \nu(-f \geq t)]\, dt - \int_0^\infty \nu(-f \geq t)\, dt \\
&= -\left(\int_{-\infty}^0 [\nu(-f \geq t) - \nu(\Omega)]\, dt + \int_0^\infty \nu(-f \geq t)\, dt\right) \\
&= -\nu_c(-f).
\end{aligned}$$

Suppose ν is balanced. Then $\nu(A) \leq \mu(A) \leq \overline{\nu}(A)$ for each $A \in \Sigma$ and each $\mu \in \mathit{core}(\nu)$. In turn this implies that, given any $f \in B(\Sigma)$, $\nu(f \geq t) \leq \mu(f \geq t) \leq \overline{\nu}(f \geq t)$ for each $t \in \mathbb{R}$. By the monotonicity of the Riemann integral,

$$\begin{aligned}
& \int_0^\infty \overline{\nu}(f \geq t)\, dt + \int_{-\infty}^0 [\overline{\nu}(f \geq t) - \overline{\nu}(\Omega)]\, dt \\
\geq & \int_0^\infty \mu(f \geq t)\, dt + \int_{-\infty}^0 [\mu(f \geq t) - \overline{\nu}(\Omega)]\, dt \\
\geq & \int_0^\infty \nu(f \geq t)\, dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(\Omega)]\, dt,
\end{aligned}$$

and so $\nu_c(f) \leq \mu(f) \leq \overline{\nu}_c(f)$, as desired. ■

In general, Choquet functionals $\nu : B(\Sigma) \to \mathbb{R}$ are not additive, that is, it is in general false that $\nu_c(f + g) = \nu_c(f) + \nu_c(g)$. However, the next result, due to Dellacherie [13], shows that additivity holds in a restricted sense. Say that two functions $f, g \in B(\Sigma)$ are *comonotonic* (short for "commonly monotonic") if $(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0$ for any pair $\omega, \omega' \in \Omega$. That is, two functions are comonotonic provided they have a similar pattern.

Theorem 26 *Suppose* $\nu : B(\Sigma) \to \mathbb{R}$ *is the Choquet functional induced by a game* $\nu \in bv(\Sigma)$*. Then,* $\nu_c(f+g) = \nu_c(f) + \nu_c(g)$ *provided* f *and* g *are comonotonic, and* $f + g \in B(\Sigma)$*.*

To prove this result we need couple of useful lemmas. The first one says that two functions f and g are comonotonic if and only if all their upper sets are nested. This is trivially true for the two collections $(f \geq t)$ and $(g \geq t)$ separately; the interesting part here is that f and g are comonotonic if and only if this is still the case for the combined collection $\{(f \geq t)\}_{t\in\mathbb{R}} \cup \{(g \geq t)\}_{t\in\mathbb{R}}$. For a proof of this lemma we refer to [17, Prop. 4.5].

Lemma 27 *Two functions* $f, g \in B(\Sigma)$ *are comonotonic if and only if the overall collection of all upper sets* $(f \geq t)$ *and* $(g \geq t)$ *is a chain.*

The next lemma says that we can replicate games over chains with suitable charges. The non-additivity of a game is, therefore, immaterial as long as we restrict ourselves to chains.

Lemma 28 *Let* $\nu \in bv(\Sigma)$*. Given any chain* $\mathcal{C}$ *in* Σ *there is* $\mu \in ba(\Sigma)$ *such that*

$$\mu(A) = \nu(A) \qquad \text{for all } A \in \mathcal{C}. \tag{16}$$

If, in addition, ν *is a capacity, then we can take* $\mu \in ba^+(\Sigma)$*.*

Proof. It is enough to prove the result for a capacity ν, as the extension to any game in $bv(\Sigma)$ is routine in view of their decomposition as differences of capacities given in Proposition 1.

Consider first a finite chain $\varnothing = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} = \Omega$. Let Σ_0 be the finite subalgebra of Σ generated by such chain. Let $\mu_0 \in ba^+(\Sigma_0)$ be defined by

$$\mu_0(A_{i+1} \backslash A_i) = \nu(A_{i+1}) - \nu(A_i) \quad \text{for } i = 1, ..., n.$$

By standard extension theorems for positive charges (see, e.g., [5, Corollary 3.3.4]), there exists $\mu \in ba^+(\Sigma)$ which extends μ_0 on Σ, i.e., $\mu(A) = \mu_0(A)$ for each $A \in \Sigma_0$. Hence, μ is the desired charge.

Now, let $\mathcal{C}$ be any chain. Let $\{\mathcal{C}_\alpha\}_\alpha$ be the collection of all its finite subchains, and set $\Gamma_\alpha = \{\mu \in ba^+(\Sigma) : \mu(A) = \nu(A) \text{ for each } A \in \mathcal{C}_\alpha\}$. By what we just proved, each Γ_α is nonempty. Moreover, the collection $\{\Gamma_\alpha\}_\alpha$ has the finite intersection property. For, let $\{\mathcal{C}_i\}_{i=1}^n \subseteq \{\mathcal{C}_\alpha\}_\alpha$ be a finite collection. Since $\bigcup_{i=1}^{n} \mathcal{C}_i$ is in turn a finite chain, by proceeding as before it is easy to establish the existence of a $\mu \in ba(\Sigma)$ such that $\mu(A) = \nu(A)$ for each $A \in \bigcup_{i=1}^{n} \mathcal{C}_i$. As $\mu \in \bigcap_{i=1}^{n} \Gamma_i$, the intersection $\bigcap_{i=1}^{n} \Gamma_i$ is nonempty, as desired.

Each Γ_α is a weak*-closed subset of the weak*-compact set

$$\left\{\mu \in ba^+(\Sigma) : \mu(\Omega) = \nu(\Omega)\right\}.$$

Since $\{\Gamma_\alpha\}_\alpha$ has the finite intersection property, we conclude that $\bigcap_\alpha \Gamma_\alpha \neq \varnothing$. Any charge $\mu \in \bigcap_\alpha \Gamma_\alpha$ satisfies (16). ■

Proof of Theorem 26. Suppose f and g are comonotonic functions in $B(\Sigma)$. Then, the sum $f+g$ is comonotonic with both f and g, so that the collection $\{f, g, f+g\}$ consists of pairwise comonotonic functions. Let

$$\mathcal{C} = \{(f \geq t)\}_{t \in \mathbb{R}} \cup \{(g \geq t)\}_{t \in \mathbb{R}} \cup \{(f+g \geq t)\}_{t \in \mathbb{R}}.$$

By Lemma 27, $\mathcal{C}$ is a chain. By Lemma 28, there is $\mu \in ba(\Sigma)$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{C}$. Hence,

$$\begin{aligned} \int f d\nu + \int g d\nu &= \int f d\mu + \int g d\mu \\ &= \int (f+g)\, d\mu = \int (f+g)\, d\nu, \end{aligned}$$

as desired. ■

As constant functions are comonotonic with all other functions, comonotonic additivity is a much stronger property than translation invariance. The next result of Bassanezi and Greco [3, Thm 2.1] shows that comonotonic additivity is actually the "best" possible type of additivity for Choquet functionals.

Proposition 29 *Suppose Σ contains all singletons. Then, two functions $f, g \in B(\Sigma)$, with $f+g \in B(\Sigma)$, are comonotonic if and only if it holds $\nu_c(f+g) = \nu_c(f) + \nu_c(g)$ for all Choquet functionals induced by convex capacities $\nu : \Sigma \to \mathbb{R}$.*

Proof. The "only if" part holds by Theorem 26. As to "if" part, assume it holds

$$\nu_c(f+g) = \nu_c(f) + \nu_c(g) \tag{17}$$

for all Choquet functionals induced by convex capacities. Suppose, *per contra*, that f and g are not comonotonic. Then, there exist $\omega', \omega'' \in \Omega$ such that $[f(\omega') - f(\omega'')][g(\omega') - g(\omega'')] < 0$. Say that $f(\omega') < f(\omega'')$ and $g(\omega') > g(\omega'')$, and consider the convex game

$$u_{\{\omega',\omega''\}}(A) = \begin{cases} 1 & \text{if } \{\omega', \omega''\} \subseteq A \\ 0 & \textit{else} \end{cases}.$$

By Example 20, $u_{\{\omega',\omega''\},c}(f) = f(\omega')$ and $u_{\{\omega',\omega''\},c}(g) = g(\omega'')$. Hence,

$$\begin{aligned} u_{\{\omega',\omega''\},c}(f+g) &= \min\{(f+g)(\omega'), (f+g)(\omega'')\} \\ &\neq f(\omega') + g(\omega'') = u_{\{\omega',\omega''\},c}(f) + u_{\{\omega',\omega''\},c}(g), \end{aligned}$$

which contradicts (17). ■

Notice that the argument used to prove the last result can be adapted to give the following characterization of comonotonicity: when Σ contains all singletons, two functions $f, g \in B(\Sigma)$ are comonotonic if and only if

$$\inf_{\omega \in A}(f(\omega) + g(\omega)) = \inf_{\omega \in A} f(\omega) + \inf_{\omega \in A} g(\omega)$$

for all $A \in \Sigma$.

Lemmas 27 and 28 are especially useful in finding counterparts for games and for their Choquet integrals of standard results that hold in the additive case. Theorem 26 is a first important example since through these lemmas we could derive the counterpart for Choquet integrals of the additivity of standard integrals. We close this subsection with another simple illustration of this feature of Lemmas 27 and 28 by showing a version for Choquet integrals of the classic Jensen inequality.

Proposition 30 *Let ν be a capacity with $\nu(\Omega) = 1$. Given a monotone convex function $\phi : \mathbb{R} \to \mathbb{R}$, for each $f \in B(\Sigma)$ the following holds:*

$$\int \phi(f)\, d\nu \geq \phi\left(\int f d\nu\right).$$

Proof. Given any $f \in B(\Sigma)$, the functions $\phi \circ f$ and f are comonotonic. By Lemmas 27 and 28, there is $\mu \in ba^+(\Sigma)$ such that $\mu(f \geq t) = \nu(f \geq t)$ and $\mu(\phi(f) \geq t) = \nu(\phi(f) \geq t)$ for each $t \in \mathbb{R}$. In turn, this implies $\mu(\Omega) = \nu(\Omega) = 1$, $\int \phi(f)\,d\nu = \int \phi(f)\,d\mu$, and $\int f d\nu = \int f d\mu$. By the standard Jensen inequality:

$$\int \phi(f)\,d\nu = \int \phi(f)\,d\mu \geq \phi\left(\int f d\mu\right) = \phi\left(\int f d\nu\right),$$

as desired. ■

4 Representation

Summing up, Choquet functionals are positively homogeneous, comonotonic additive, and Lipschitz continuous; they are also monotone provided the underlying game does.

A natural question is whether these properties actually characterize Choquet functionals among all the functionals defined on $B(\Sigma)$. Schmeidler [56] showed that this the case and we now present his result.

Theorem 31 *Let $\widetilde{\nu} : B(\Sigma) \to \mathbb{R}$ be a functional. Define the game $\nu(A) = \widetilde{\nu}(1_A)$ on Σ. The following conditions are equivalent:*

(i) $\widetilde{\nu}$ is monotone and comonotonic additive;

(ii) ν is a capacity and, for all $f \in B(\Sigma)$, it holds:

$$\widetilde{\nu}(f) = \int_0^\infty \nu(f \geq t)\,dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(\Omega)]\,dt. \qquad (18)$$

Remarks. (i) Positive homogeneity is a redundant condition here as it is implied by comonotonic additivity and monotonicity, as shown in the proof. (ii) Zhou [71] proved a version of this result on Stone lattices.

Proof. (ii) trivially implies (i). Conversely, assume (i). We divide the proof into three steps.

Step 1. For any $f \in B(\Sigma)$ and any integer n, by comonotonic additivity we have $\widetilde{\nu}(f) = \widetilde{\nu}\left(n\frac{f}{n}\right) = n\widetilde{\nu}\left(\frac{f}{n}\right)$. Namely, $\widetilde{\nu}\left(\frac{f}{n}\right) = \frac{1}{n}\widetilde{\nu}(f)$. Hence, given any positive rational number $\alpha = m/n$,

$$\widetilde{\nu}\left(\frac{m}{n}f\right) = \widetilde{\nu}\left(\frac{f}{n} + \cdots + \frac{f}{n}\right) = \widetilde{\nu}\left(\frac{f}{n}\right) + \cdots + \widetilde{\nu}\left(\frac{f}{n}\right) = \frac{m}{n}\widetilde{\nu}(f).$$

As a result, we have $\widetilde{\nu}(\lambda f) = \lambda\widetilde{\nu}(f)$ for any $\lambda \in \mathbb{Q}_+$. In particular, this implies $0 = \widetilde{\nu}(\lambda 1_\Omega - \lambda 1_\Omega) = \lambda\nu(\Omega) + \widetilde{\nu}(-\lambda 1_\Omega)$ for each $\lambda \in \mathbb{Q}_+$, and so $\widetilde{\nu}(f + \lambda 1_\Omega) = \widetilde{\nu}(f) + \widetilde{\nu}(\lambda 1_\Omega) = \widetilde{\nu}(f) + \lambda\nu(\Omega)$ for each $f \in B(\Sigma)$ and each $\lambda \in \mathbb{Q}$.

Step 2. We now prove that $\widetilde{\nu}$ is supnorm continuous. Let $f, g \in B(\Sigma)$ and let $\{r_n\}_n$ be a sequence of rationals such that $r_n \downarrow \|f - g\|$. As $f \leq g + \|f - g\| \leq g + r_n$, it follows that $\widetilde{\nu}(f) \leq \widetilde{\nu}(g) + r_n\nu(\Omega)$. Consequently, $|\widetilde{\nu}(f) - \widetilde{\nu}(g)| \leq r_n\nu(\Omega)$. As $n \to \infty$, we get $|\widetilde{\nu}(f) - \widetilde{\nu}(g)| \leq \|f - g\|\,\nu(\Omega)$. Hence, $\widetilde{\nu}$ is Lipschitz continuous, and so supnorm continuous.

In turn, this implies $\widetilde{\nu}(\lambda f) = \lambda\widetilde{\nu}(f)$ for all $\lambda \geq 0$ and $\widetilde{\nu}(f + \lambda 1_\Omega) = \widetilde{\nu}(f) + \lambda\nu(\Omega)$ for each $f \in B(\Sigma)$ and each $\lambda \in \mathbb{R}$, i.e., $\widetilde{\nu}$ is translation invariant.

Step 3. It remains to show that (18) holds, i.e., that $\widetilde{\nu}(f) = \nu_c(f)$ for all $f \in B(\Sigma)$. Since both $\widetilde{\nu}$ and ν_c are supnorm continuous and $B_0(\Sigma)$ is supnorm dense in $B(\Sigma)$, it is enough to show that $\widetilde{\nu}(f) = \nu_c(f)$ for all $f \in B_0(\Sigma)$.

Let $f \in B_0(\Sigma)$. Since both $\widetilde{\nu}$ and ν_c are translation invariant, it is enough to show that $\widetilde{\nu}(f) = \nu_c(f)$ for $f \geq 0$. As $f \in B_0(\Sigma)$, we can write $f = \sum_{i=1}^k \alpha_i 1_{A_i}$, where $\{A_i\}_{i=1}^k \subseteq \Sigma$ is a suitable partition of Ω and $\alpha_1 > \cdots > \alpha_k$. Setting $D_i = \bigcup_{j=1}^i A_j$ and $\alpha_{k+1} = 0$, we can then write $f = \sum_{i=1}^{k-1}(\alpha_i - \alpha_{i+1})\,1_{D_i} + \alpha_k 1_\Omega$. As the functions $\{(\alpha_i - \alpha_{i+1})\,1_{D_i}\}_{i=1}^{k-1}$ and $\alpha_k 1_\Omega$ are pairwise comonotonic, by the comonotonic additivity and positive homogeneity of $\widetilde{\nu}$ we have

$$\widetilde{\nu}(f) = \sum_{i=1}^{k-1}(\alpha_i - \alpha_{i+1})\,\nu\left(\bigcup_{j=1}^i A_j\right) + \alpha_k 1_\Omega.$$

Since $\sum_{i=1}^k (\alpha_i - \alpha_{i+1})\,\nu\left(\bigcup_{j=1}^i A_j\right) = \int_0^\infty \nu(f \geq t)\,dt$, we conclude that $\widetilde{\nu}(f) = \int_0^\infty \nu(f \geq t)\,dt$, i.e., $\widetilde{\nu}(f) = \nu_c(f)$, as desired. ■

Next we extend Schmeidler's Theorem to the non-monotonic case. Given a functional $\widetilde{\nu} : B(\Sigma) \to \mathbb{R}$ and any two $f, g \in B(\Sigma)$ with $f \leq g$, set

$$V(f; g) = \sup \sum_{i=0}^{n-1} |\widetilde{\nu}(f_{i+1}) - \widetilde{\nu}(f_i)|,$$

where the supremum is taken over all finite chains $f = f_0 \leq f_1 \leq \cdots \leq f_n = g$. We say that $\widetilde{\nu}$ is of *bounded variation* if $V(0; f) < +\infty$ for all $f \in B^+(\Sigma)$.

Theorem 32 *Let $\widetilde{\nu} : B(\Sigma) \to \mathbb{R}$ be a functional. Define the game $\nu(A) = \widetilde{\nu}(1_A)$ on Σ. The following conditions are equivalent:*

(i) $\widetilde{\nu}$ is comonotonic additive and of bounded variation;

(ii) $\widetilde{\nu}$ is comonotonic additive and supnorm continuous on $B^+(\Sigma)$, and $\nu \in bv(\Sigma)$;

(iii) $\nu \in bv(\Sigma)$ and, for all $f \in B(\Sigma)$,

$$\widetilde{\nu}(f) = \int_0^\infty \nu(f \geq t)\, dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(\Omega)]\, dt.$$

Remark. When Σ is finite, the requirement $\nu \in bv(\Sigma)$ becomes superfluous in conditions (ii) and (iii) as all finite games are of bounded variation.

Before proving the result, we give a useful lemma. Observe that the decomposition $f = (f-t)^+ + (f \wedge t)$ reduces to the standard $f = f^+ - f^-$ when $t = 0$.

Lemma 33 *Let $\widetilde{\nu} : B(\Sigma) \to \mathbb{R}$ be a comonotonic additive functional. Then,*

$$\widetilde{\nu}(f) = \widetilde{\nu}\left((f-t)^+\right) + \widetilde{\nu}(f \wedge t) \qquad \text{for each } t \in \mathbb{R} \text{ and } f \in B(\Sigma).$$

Proof. Given any $t \in \mathbb{R}$, the functions $(f-t)^+$ and $f \wedge t$ are comonotonic. In fact, for any $\omega, \omega' \in \Omega$ we have:

$$\begin{aligned} & \left[(f-t)^+(\omega) - (f-t)^+(\omega')\right] \left[(f \wedge t)(\omega) - (f \wedge t)(\omega')\right] \\ = \; & (f-t)^+(\omega)(f \wedge t)(\omega) - (f-t)^+(\omega)(f \wedge t)(\omega') \\ & - (f-t)^+(\omega')(f \wedge t)(\omega) + (f-t)^+(\omega')(f \wedge t)(\omega') \\ = \; & (f-t)^+(\omega)(f-t)^-(\omega') + (f-t)^+(\omega')(f-t)^-(\omega) \geq 0, \end{aligned}$$

as desired. ■

Proof of Theorem 32. (i) implies (ii). Clearly, $\nu \in bv(\Sigma)$. We want to show that (i) implies that $\widetilde{\nu}$ is supnorm continuous over $B^+(\Sigma)$. As Step 1 of the proof of Theorem 31 still holds here, we have $\widetilde{\nu}(f + \lambda 1_\Omega) = \widetilde{\nu}(f) + \lambda\nu(\Omega)$ for each $f \in B(\Sigma)$ and each $\lambda \in \mathbb{Q}$. That is, $\widetilde{\nu}$ is translation invariant w.r.t. $\mathbb{Q}$.

Let $f, g \in B(\Sigma)$ with $f \leq g$. If $f \geq 0$, then $V(f; g) \leq V(0; g) < +\infty$. Suppose f is not necessarily positive. There exists $\lambda \in \mathbb{Q}_+$ such that $f + \lambda \geq 0$ and $g + \lambda \geq 0$. By the translation invariance w.r.t. $\mathbb{Q}$ of $\widetilde{\nu}$, we have $V(f; g) = V(f + \lambda; g + \lambda)$ for all $\lambda \in \mathbb{Q}$. Hence, $V(f; g) = V(f + \lambda; g + \lambda) < +\infty$.

It is easy to see that $V(0; \lambda f) = \lambda V(0; f)$ for all $\lambda \in \mathbb{Q}_+$. The next claim gives a deeper property of $V(f; g)$.

Claim. For all $f \geq 0$ and all $\lambda \in \mathbb{Q}_+$, it holds

$$V(-\lambda; f) = V(-\lambda; 0) + V(0; f).$$

Proof of the Claim. If $f \leq h \leq g$, we have $V(f; g) \geq V(f; h) + V(h; g)$. Hence, it suffices to show that $V(-\lambda; f) \leq V(-\lambda; 0) + V(0; f)$.

By definition, for any $\varepsilon > 0$ there exists a chain $\{\varphi_i\}_{i=0}^n$ such that

$$\sum_{i=0}^{n-1} \left|\widetilde{\nu}(\varphi_{i+1}) - \widetilde{\nu}(\varphi_i)\right| \geq V(-\lambda; f) - \varepsilon,$$

with $\varphi_0 = -\lambda$ and $\varphi_n = f$. For each φ_i, consider the two functions $\varphi_i^- = -(\varphi_i \wedge 0)$ and $\varphi_i^+ = \varphi_i \vee 0$ and the two chains $\{-\varphi_i^-\}$ and $\{\varphi_i^+\}$. The former chain is relative to $V(-\lambda; 0)$, while the latter is relative to $V(0; f)$. Therefore, we have

$$\begin{aligned} & V(-\lambda; 0) + V(0; f) \qquad (19) \\ \geq & \sum_{i=0}^{n-1} \left|\widetilde{\nu}(-\varphi_{i+1}^-) - \widetilde{\nu}(-\varphi_i^-)\right| + \sum_{i=0}^{n-1} \left|\widetilde{\nu}(\varphi_{i+1}^+) - \widetilde{\nu}(\varphi_i^+)\right| \\ = & \sum_{i=0}^{n-1} \left(\left|\widetilde{\nu}(-\varphi_{i+1}^-) - \widetilde{\nu}(-\varphi_i^-)\right| + \left|\widetilde{\nu}(\varphi_{i+1}^+) - \widetilde{\nu}(\varphi_i^+)\right|\right). \end{aligned}$$

On the other hand, by Lemma 33 for each i we have $\widetilde{\nu}(\varphi_i) = \widetilde{\nu}(\varphi_i^+) + \widetilde{\nu}(-\varphi_i^-)$, and so:

$$\begin{aligned}\left|\widetilde{\nu}(\varphi_{i+1}) - \widetilde{\nu}(\varphi_i)\right| &= \left|\widetilde{\nu}(\varphi_{i+1}^+) + \widetilde{\nu}(-\varphi_{i+1}^-) - \widetilde{\nu}(\varphi_i^+) - \widetilde{\nu}(-\varphi_i^-)\right| \\ &\leq \left|\widetilde{\nu}(\varphi_{i+1}^+) - \widetilde{\nu}(\varphi_i^+)\right| + \left|\widetilde{\nu}(-\varphi_{i+1}^-) - \widetilde{\nu}(-\varphi_i^-)\right|.\end{aligned}$$

In view of (19), we can write

$$V(-\lambda; 0) + V(0; f) \geq \sum_{i=0}^{n-1} \left|\widetilde{\nu}(\varphi_{i+1}) - \widetilde{\nu}(\varphi_i)\right| \geq V(-\lambda; f) - \varepsilon,$$

which proves our claim.

Define the monotone functional $\widetilde{\nu}_1(f) = V(0; f)$ on $B^+(\Sigma)$. For each $\lambda \in \mathbb{Q}_+$ we have

$$\begin{aligned}\widetilde{\nu}_1(f+\lambda) &= V(0; f+\lambda) = V(-\lambda; f) = V(-\lambda; 0) + V(0; f) \\ &= V(0; \lambda) + V(0; f) = \lambda V(0; 1) + V(0; f) = \lambda \widetilde{\nu}_1(1_\Omega) + \widetilde{\nu}_1(f).\end{aligned}$$

Hence, $\widetilde{\nu}_1$ is translation invariant w.r.t. $\mathbb{Q}_+$. Since $\widetilde{\nu}_1$ is monotone, by Step 2 of the proof of Theorem 31 it is Lipschitz continuous, and so supnorm continuous.

Consider the functional $\widetilde{\nu}_2 = \widetilde{\nu}_1 - \widetilde{\nu}$ on $B^+(\Sigma)$. The functional $\widetilde{\nu}_2$ is monotone; moreover, it is translation invariant w.r.t. $\mathbb{Q}$ as both $\widetilde{\nu}_1$ and $\widetilde{\nu}$ do. Consequently, by Step 2 of the proof of Theorem 31 $\widetilde{\nu}_2$ is supnorm continuous. As $\widetilde{\nu} = \widetilde{\nu}_1 - \widetilde{\nu}_2$, we conclude that also $\widetilde{\nu}$ is supnorm continuous, thus completing the proof that (i) implies (ii).

(ii) implies (iii). Step 1 of the proof of Theorem 31 holds here as well. Hence, $\widetilde{\nu}(\lambda f) = \lambda \widetilde{\nu}(f)$ for all $\lambda \in \mathbb{Q}_+$, and $\widetilde{\nu}(f + \lambda 1_\Omega) = \widetilde{\nu}(f) + \lambda \nu(\Omega)$ for each $f \in B(\Sigma)$ and each $\lambda \in \mathbb{Q}$. By supnorm continuity, $\widetilde{\nu}(\lambda f) = \lambda \widetilde{\nu}(f)$ for all $\lambda \geq 0$, and $\widetilde{\nu}(f + \lambda 1_\Omega) = \widetilde{\nu}(f) + \lambda \nu(\Omega)$ for each $\lambda \in \mathbb{R}$. The functional $\widetilde{\nu}$ is, therefore, positively homogeneous and translation invariant.

Let ν_c be the Choquet functional associated with ν. As $\nu \in bv(\Sigma)$, ν_c is well defined and supnorm continuous. We want to show that $\widetilde{\nu} = \nu_c$. Since both $\widetilde{\nu}$ and ν_c are supnorm continuous and $B_0(\Sigma)$ is supnorm dense in $B(\Sigma)$, it is enough to show that $\widetilde{\nu}(f) = \nu_c(f)$ for each $f \in B_0(\Sigma)$. This can be established by proceeding as in Step 3 of the proof of Theorem 31.

(iii) implies (i). It remains to show that the Choquet functional ν_c is of bounded variation as long as $\nu \in bv(\Sigma)$. By Proposition 1, there exist capacities ν^1 and ν^2 such that $\nu = \nu^1 - \nu^2$. Hence, $\nu_c = \nu_c^1 - \nu_c^2$ and so the functional ν_c is the difference of two monotone functionals. This implies

$$V(f; g) \leq \nu_c^1(g) - \nu_c^1(f) + \nu_c^2(g) - \nu_c^2(f),$$

and we conclude that ν_c is of bounded variation. ■

5 Convex Games

Convex games are an interesting class of games and played an important role in Schmeidler's approach to ambiguity, as explained in chapter 1. Here we show some of their remarkable mathematical properties.

We begin by proving formally that convexity can be formulated as in Eq. (1), a version useful in game theory for interpreting supermodularity in terms of marginal values (see [44]).

Proposition 34 *For any game ν, the following properties are equivalent:*

(i) ν is convex;

(ii) for all sets A, B, and C such that $A \subseteq B$ and $B \cap C = \varnothing$,

$$\nu(A \cup C) - \nu(A) \leq \nu(B \cup C) - \nu(B);$$

(iii) for all disjoint sets A, B, and C:

$$\nu(B \cup A) - \nu(B) \leq \nu(B \cup C \cup A) - \nu(B \cup C).$$

Proof. (ii) easily implies (iii). Assume (ii) holds. Since $(A \cup B) \setminus A = B \setminus (A \cap B)$, to check the supermodularity of ν is enough to set $C = (A \cup B) \setminus A$. Finally, assume (i) holds. If the sets A, B, and C are disjoint, then $(B \cup C) \cap (B \cup A) = B$, and so supermodularity implies (iii), as desired. ■

The next result, due to Choquet [11, p. 289], shows that the convexity of the game and the superlinearity of the associated Choquet functional are two faces of the same coin.[7] Recall that, by Proposition 15, $B(\Sigma)$ is a lattice and it becomes a vector lattice when Σ is σ-algebra.

[7] A functional is superlinear if it is positively homogeneous and superadditive. Recall that, by Proposition 24, Choquet functionals are always positively homogeneous.

Theorem 35 *For any game ν in $bv(\Sigma)$, the following conditions are equivalent:*

(i) *ν is convex,*

(ii) *ν_c is superadditive on $B(\Sigma)$, i.e., $\nu_c(f+g) \geq \nu_c(f) + \nu_c(g)$ for all $f, g \in B(\Sigma)$ such that $f + g \in B(\Sigma)$.*

(iii) *ν_c is supermodular on $B(\Sigma)$, i.e., $\nu_c(f \vee g) + \nu_c(f \wedge g) \geq \nu_c(f) + \nu_c(g)$ for all $f, g \in B(\Sigma)$.*

Proof. We prove that both (ii) and (iii) are equivalent to (i).

(i) implies (ii). Given $f \in B^+(\Sigma)$ and $E \in \Sigma$, we have:

$$(f + 1_E \geq t) = (f \geq t) \cup (E \cap (f \geq t-1)),$$

and so $f + 1_E \in B^+(\Sigma)$. In turn, this implies $f + g \in B^+(\Sigma)$ whenever $g \in B^+(\Sigma)$ is simple. Moreover, as ν is convex, we get

$$\nu(f + 1_E \geq t) \geq \nu(f \geq t) + \nu(E \cap (f \geq t-1)) - \nu(E \cap (f \geq t)).$$

Consequently,

$$\begin{aligned}
& \nu_c(f + 1_E) \\
= & \int_0^\infty \nu(f + 1_E \geq t)\, dt \\
\geq & \int_0^\infty \nu(f \geq t)\, dt + \int_0^\infty \nu(E \cap (f \geq t-1))\, dt - \int_0^\infty \nu(E \cap (f \geq t))\, dt \\
= & \nu_c(f) + \int_{-1}^0 \nu(E \cap (f \geq t))\, dt = \nu_c(f) + \nu(E).
\end{aligned}$$

As ν_c is positive homogeneous, for each $\lambda \geq 0$ we have:

$$\begin{aligned}
\nu_c(f + \lambda 1_E) &= \lambda \nu_c\left(\frac{f}{\lambda} + 1_E\right) \geq \lambda\left(\nu_c\left(\frac{f}{\lambda}\right) + \nu(E)\right) \\
&= \nu_c(f) + \lambda \nu(E).
\end{aligned}$$

Let $g \in B^+(\Sigma)$ be a simple function. We can write $g = \sum_{i=1}^n \lambda_i 1_{D_i}$, where $D_1 \subseteq \cdots \subseteq D_n$ and $\lambda_i \geq 0$ for each $i = 1, ..., n$. As g is simple, we have

$f+g \in B^{+}(\Sigma)$. Hence,

$$\begin{aligned}
\nu_c(f+g) &= \nu_c\left(f+\sum_{i=1}^{n}\lambda_i 1_{D_i}\right) \geq \nu_c\left(f+\sum_{i=2}^{n}\lambda_i 1_{D_i}\right)+\lambda_1\nu(D_1) \\
&\geq \cdots \\
&\geq \nu_c(f)+\sum_{i=1}^{n}\lambda_i\nu(D_i) = \nu_c(f)+\nu_c(g),
\end{aligned}$$

as desired. To show that the inequality $\nu(f+g) \geq \nu(f)+\nu(g)$ holds for all $f, g \in B(\Sigma)$ it is now enough to use the translation invariance and supnorm continuity of ν_c.

(ii) implies (i). Given any sets A and B, it holds

$$1_{A\cup B}+1_{A\cap B} = 1_A + 1_B.$$

Since the characteristic functions $1_{A\cup B}$ and $1_{A\cap B}$ are comonotonic, we then have:

$$\begin{aligned}
\nu(A\cup B)+\nu(A\cap B) &= \nu_c(1_{A\cup B})+\nu_c(1_{A\cap B}) = \nu_c(1_{A\cup B}+1_{A\cap B}) \\
&= \nu_c(1_A+1_B) \geq \nu_c(1_A)+\nu_c(1_B) = \nu(A)+\nu(B),
\end{aligned}$$

and so the game ν is convex, as desired.

(i) implies (iii). As ν_c is translation invariant, it is enough to prove the implication for f and g positive. It is easy to check that, for each $t \in \mathbb{R}$, it holds:

$$\begin{aligned}
(f\vee g \geq t) &= (f \geq t)\cup(g \geq t) \\
(f\wedge g \geq t) &= (f \geq t)\cap(g \geq t).
\end{aligned}$$

Therefore, if ν is convex, then

$$\nu(f\vee g \geq t)+\nu(f\wedge g \geq t) \geq \nu(f \geq t)+\nu(g \geq t).$$

Hence,

$$\begin{aligned}
\nu_c(f\vee g)+\nu_c(f\wedge g) &= \int_0^\infty \nu(f\vee g \geq t)\,dt+\int_0^\infty \nu(f\wedge g \geq t)\,dt \\
&= \int_0^\infty [\nu(f\vee g \geq t)+\nu(f\wedge g \geq t)]\,dt \\
&\geq \int_0^\infty [\nu(f \geq t)+\nu(g \geq t)]\,dt = \nu_c(f)+\nu_c(g),
\end{aligned}$$

as desired.

(iii) implies (i). We have $1_A \vee 1_B = 1_{A\cup B}$ and $1_A \wedge 1_B = 1_{A\cap B}$. Hence, if we put $f = 1_A$ and $g = 1_B$ in the inequality $\nu_c(f \vee g) + \nu_c(f \wedge g) \geq \nu_c(f) + \nu_c(g)$, we get $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$, as desired. ■

By Theorem 35, a game is convex if and only if the associated Choquet functional ν_c is superlinear, that is, superadditive and positively homogeneous. This is a useful property that, for example, makes it possible to use the classic Hahn-Banach Theorem in studying convex games.

In order to do so, however, we first have to deal with a technical problem: unless Σ is a σ-algebra, the space $B(\Sigma)$ is not in general a vector space, something needed to apply the Hahn-Banach Theorem and other standard functional analytic results. There are at least two ways to bypass the problem. The first one is to consider the vector space $B_0(\Sigma)$ of Σ-measurable simple functions in place of the whole set $B(\Sigma)$. This can be enough as long as one is interested in using results that, like the Hahn-Banach Theorem, hold on any vector space. There are important results, however, that only hold on Banach spaces (e.g., the Uniform Boundedness Principle). In this case $B_0(\Sigma)$, which is not a Banach space, is useless.

A solution is to consider $\overline{B}(\Sigma)$, the supnorm closure $\overline{B}(\Sigma)$ of $B_0(\Sigma)$,[8] which is a Banach lattice under the supnorm ([20, p. 258]). $B(\Sigma)$ is a dense subset of $\overline{B}(\Sigma)$; it holds $B(\Sigma) = \overline{B}(\Sigma)$ when Σ is a σ-algebra, and so in this case $B(\Sigma)$ itself is a Banach lattice. If Σ is not a σ-algebra, to work with the Banach lattice $\overline{B}(\Sigma)$ we have to extend on it the Choquet functional ν_c, which is originally defined on $B(\Sigma)$.

Lemma 36 *Any Choquet functional $\nu_c : B(\Sigma) \to \mathbb{R}$ induced by a game $\nu \in bv(\Sigma)$ admits a unique supnorm continuous extension on $\overline{B}(\Sigma)$. Such extension is positively homogenous and comonotonic additive.*

Proof. By Proposition 24(iv), ν_c is Lipschitz continuous on $B(\Sigma)$. By standard results ([1, p. 77]), it then admits a unique supnorm continuous extension on the closure $\overline{B}(\Sigma)$. Using its supnorm continuity, such extension is easily seen to be positively homogeneous. As to comonotonic additivity, we first prove the following claim.

[8] That is, $f \in \overline{B}(\Sigma)$ provided there is a sequence $\{f_n\}_n \subseteq B_0(\Sigma)$ such that $\lim_n \|f - f_n\| = 0$. Here we are viewing $B_0(\Sigma)$ as a subset of the set of all bounded functions $f : \Omega \to \mathbb{R}$.

Claim. Given any two comonotonic and supnorm bounded functions f and g, there exist two sequences of simple functions $\{f_n\}_n$ and $\{g_n\}_n$ uniformly converging to f and g, respectively, and such that f_n and g_n are comonotonic for each n.

Proof of the Claim. It is enough to prove the claim for positive functions. Let $f : \Omega \to \mathbb{R}$ be positive and supnorm bounded, so that there exists a constant $M > 0$ such that $0 \leq f(\omega) \leq M$ for each $\omega \in \Omega$. Let $M = \alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 = 0$, with $\alpha_i = (i/n) M$ for each $i = 0, 1, ..., n$. Set $A_i = (f \geq \alpha_i)$ for each $i = 1, ..., n-1$, and define $f_n : \Omega \to \mathbb{R}$ as $f_n = \sum_{i=1}^{n-1} \alpha_i 1_{A_i}$. The collection of upper sets $\{(f_n \geq t)\}_{t \in \mathbb{R}}$ is included in $\{(f \geq t)\}_{t \in \mathbb{R}}$ and $\|f - f_n\| = \max_{i \in \{0,...,n-1\}} (\alpha_{i+1} - \alpha_i) = M/n$.

In a similar way, for each n we can construct a simple function g_n such that the collection of upper sets $\{(g_n \geq t)\}_{t \in \mathbb{R}}$ is included in $\{(g \geq t)\}_{t \in \mathbb{R}}$ and $\|g - g_n\| = M/n$. By Lemma 27, the collections $\{(g \geq t)\}_{t \in \mathbb{R}}$ and $\{(f \geq t)\}_{t \in \mathbb{R}}$ together form a chain. Hence, by what we just proved, for each n the collections $\{(g \geq t)\}_{t \in \mathbb{R}}$, $\{(g_n \geq t)\}_{t \in \mathbb{R}}$, $\{(f \geq t)\}_{t \in \mathbb{R}}$, and $\{(f_n \geq t)\}_{t \in \mathbb{R}}$ together form a chain as well. Again by Lemma 27, f_n and g_n are then comonotonic functions, and so the sequences $\{f_n\}_n$ and $\{g_n\}_n$ we have constructed have the desired properties. This completes the proof of the Claim.

Let $f, g \in \overline{B}(\Sigma)$. Consider the sequences $\{f_n\}_n$ and $\{g_n\}_n$ of simple functions given by the Claim. As such sequences belong to $B(\Sigma)$, by the supnorm continuity of ν_c we have:

$$\nu_c(f+g) = \lim_n \nu_c(f_n + g_n) = \lim_n \nu_c(f_n) + \lim_n \nu_c(g_n) = \nu_c(f) + \nu_c(g),$$

as desired. ■

It is convenient to denote this extension still by ν_c, and in the sequel we will write $\nu_c : \overline{B}(\Sigma) \to \mathbb{R}$. In the enlarged domain $\overline{B}(\Sigma)$ the following cleaner version of Theorem 35 holds. As $\overline{B}(\Sigma)$ is a vector space, here we can consider concavity and quasi-concavity. The latter property is the only non-trivial feature of the next result relative to Theorem 35.[9]

[9] The equivalence between the convexity of ν and the concavity of ν_c established in Corollary 37 is also a curious terminological phenomenon, which may give rise to some confusion. A simple way to avoid any problem is to use the terminology "supermodular games."

Corollary 37 *For any game ν in $bv(\Sigma)$, the following conditions are equivalent:*

(i) ν *is convex,*

(ii) ν_c *is superlinear on* $\overline{B}(\Sigma)$*,*

(iii) ν_c *is supermodular on* $\overline{B}(\Sigma)$*,*

(iv) ν_c *is concave on* $\overline{B}(\Sigma)$*,*

(v) ν_c *is quasi-concave on* $\overline{B}(\Sigma)$*, provided* $\nu(\Omega) \neq 0$*.*

Proof. In view of Theorem 35, the only nontrivial part is to show that (v) implies (iv). We will actually prove the stronger result that (iv) is equivalent to the convexity of the cone $\{f : \nu_c(f) \geq 0\}$.

Set $K = \left\{f \in \overline{B}(\Sigma) : \nu_c(f) \geq 0\right\}$. Given two functions $f, g \in \overline{B}(\Sigma)$, we have

$$\nu_c\left(f - \frac{\nu_c(f)}{\nu(\Omega)}1_\Omega\right) = 0, \;\; \nu_c\left(g - \frac{\nu_c(g)}{\nu(\Omega)}1_\Omega\right) = 0.$$

Hence, both $f - \frac{\nu_c(f)}{\nu(\Omega)}1_\Omega$ and $g - \frac{\nu_c(g)}{\nu(\Omega)}1_\Omega$ lie in K. By the convexity of K, taken $\alpha \in [0,1]$ and $\overline{\alpha} \equiv 1 - \alpha$, we have

$$\alpha f - \alpha\frac{\nu_c(f)}{\nu(\Omega)}1_\Omega + \overline{\alpha}g - \overline{\alpha}\frac{\nu_c(g)}{\nu(\Omega)}1_\Omega \in K.$$

Namely,

$$\begin{aligned} & \nu_c\left(\alpha f - \alpha\frac{\nu_c(f)}{\nu(\Omega)}1_\Omega + \overline{\alpha}g - \overline{\alpha}\frac{\nu_c(g)}{\nu(\Omega)}1_\Omega\right) \\ = \;& \nu_c(\alpha f + \overline{\alpha}g) - \alpha\nu_c(f) - \overline{\alpha}\nu_c(g) \geq 0. \end{aligned}$$

Therefore, ν_c is concave. ■

Remarks. (i) Dual properties hold for submodular games. For example, a game ν is submodular if and only if its Choquet functional ν_c is convex on $\overline{B}(\Sigma)$; equivalently, a game ν is convex if and only if its dual Choquet functional $\overline{\nu}_c$ is convex on $\overline{B}(\Sigma)$. For brevity, we omit these dual properties. (ii) Condition $\nu(\Omega) \neq 0$ in point (v) is needed. Consider the game ν on $\Omega =$

$\{\omega_1, \omega_2\}$ with $\nu(\omega_1) = 2$, $\nu(\omega_2) = -1$, and $\nu(\Omega) = 0$. Being subadditive, ν is not convex. On the other hand, its Choquet integral is

$$\nu_c(x_1, x_2) = \begin{cases} 2(x_1 - x_2) & x_1 \geq x_2 \\ x_1 - x_2 & x_2 > x_1 \end{cases},$$

which is quasi-concave.

The next result is a first consequence of the use of functional analytic tools in the study of convex games. The equivalence between (i) and (v) is due to Schmeidler [56] for positive games and to De Waegenaere and Wakker [19] for finite games; for the other equivalences we refer to Delbaen [12] and Marinacci and Montrucchio [42].

Theorem 38 *For a bounded game ν, the following conditions are equivalent:*

(i) ν is convex,

(ii) for any $A \subseteq B$ there is $\mu \in core(\nu)$ such that $\mu(A) = \nu(A)$ and $\mu(B) = \nu(B)$.

(iii) for any finite chain $\{A_i\}_{i=1}^n$, there is $\mu \in core(\nu)$ such that $\mu(A_i) = \nu(A_i)$ for all $i = 1, ..., n$.

(iv) $\nu \in bv(\Sigma)$ and, for any chain $\{A_i\}_{i \in I}$, there is $\mu \in ext(core(\nu))$ such that $\mu(A_i) = \nu(A_i)$ for all $i \in I$.

(v) $\nu \in bv(\Sigma)$ and $\nu_c(f) = \min_{\mu \in core(\nu)} \int f d\mu$ for all $f \in \overline{B}(\Sigma)$.

(vi) $\nu_c(f) = \min_{\mu \in core(\nu)} \int f d\mu$ for all $f \in B_0(\Sigma)$.

This theorem has a few noteworthy features. First, it shows that bounded and convex games belong to $bv(\Sigma)$, so that they always have well defined Choquet integrals on $B(\Sigma)$. Second, it improves Lemma 28 by showing that in the convex case the "replicating" measures over chains can be assumed to be in the core. Finally, Theorem 38 shows that Choquet functionals of convex games can be viewed as lower envelopes of the linear functional on $B(\Sigma)$ induced by the measures in the cores. In other words, convex games are exact games of a special type, in which the close connection between the game and the measures in the core holds on the entire space $B(\Sigma)$, and not just on Σ.

Proof. The proof proceeds as follows:

$$(i) \Longrightarrow (vi) \Longrightarrow (iv) \Longrightarrow (v) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i).$$

(i) implies (vi). Given any $f \in B_0(\Sigma)$, the Choquet integral $\int f d\nu$ is well defined since $\nu \in bv(\Sigma_f)$, where Σ_f is the finite algebra generated by f. Hence, the Choquet functional $\nu_c : B_0(\Sigma) \to \mathbb{R}$ exists on the vector space $B_0(\Sigma)$, and it is positively homogeneous and translation invariant.

Let $f, g : \Omega \to \mathbb{R}$ be any two functions in $B_0(\Sigma)$. Let $\Sigma_{f,g}$ be the smallest algebra that makes both f and g measurable. As $\Sigma_{f,g}$ is finite, $\nu \in bv(\Sigma_{f,g})$ and so we can apply Theorem 35 to the restricted Choquet integral $\nu_c : B(\Sigma_{f,g}) \to \mathbb{R}$. Thus, $\nu_c(f+g) \geq \nu_c(f) + \nu_c(g)$. Since f and g were arbitrary elements of $B_0(\Sigma)$, we conclude that $\nu_c : B_0(\Sigma) \to \mathbb{R}$ is a superlinear functional on $B_0(\Sigma)$.

Let $f \in B_0(\Sigma)$. The algebraic dual of $B_0(\Sigma)$ is the space $fa(\Sigma)$ of all finitely additive games on Σ.[10] As $\nu_c : B_0(\Sigma) \to \mathbb{R}$ is superlinear, by the Hahn-Banach Theorem there is $\mu_c \in fa(\Sigma)$ such that $\mu_c(f) = \nu_c(f)$ and $\mu_c(g) \geq \nu_c(g)$ for each $g \in B_0(\Sigma)$. In other words,

$$\nu_c(f) = \min_{\mu \in C} \mu_c(f),$$

where $C = \{\mu \in fa(\Sigma) : \mu_c(f) \geq \nu_c(f) \text{ for each } f \in B_0(\Sigma)\}$. Next we show that C coincides with the set

$$C' = \{\mu \in fa(\Sigma) : \mu \geq \nu \text{ and } \mu(\Omega) = \nu(\Omega)\}.$$

Let $\mu \in C$. Then, $\mu(A) = \mu_c(1_A) \geq \nu_c(1_A) = \nu(A)$ for all $A \in \Sigma$; moreover, $-\mu(\Omega) = \mu_c(-1_\Omega) \geq \nu_c(-1_\Omega) = -\nu(\Omega)$. Hence, $\mu \in C'$. Conversely, suppose $\mu \in C'$. As $\mu \geq \nu$ and $\mu(\Omega) = \nu(\Omega)$, the definition of Choquet integral immediately implies that $\nu_c(f) \geq \mu(f)$. Hence, $\mu \in C$.

It remains to show that $C' = core(\nu)$. As $ba(\Sigma) \subseteq fa(\Sigma)$, $core(\nu) \subseteq C'$. As to the converse inclusion, suppose $\mu \in C'$. Since ν is bounded, for each $\mu \in C'$ we have $|\mu(A)| \leq 2\sup_{A \in \Sigma} |\nu(A)|$ (see Proposition 3). Then, $\mu \in ba(\Sigma)$ (see [20, p. 97]) and we conclude that $C' \subseteq core(\nu)$, as desired.

(vi) implies (iv). Consider first a finite chain $A_1 \subseteq ... \subseteq A_n$. By (vi), there exists $\mu \in core(\nu)$ such that

$$\mu_c\left(\sum_{i=1}^{n} 1_{A_i}\right) = \nu_c\left(\sum_{i=1}^{n} 1_{A_i}\right).$$

[10]Notice that $ba(\Sigma)$ is the subspace of $fa(\Sigma)$ consiting of all bounded charges.

By comonotonic additivity, $\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \nu(A_i)$. As $\mu \in core(\nu)$, we have $\mu(A_i) \geq \nu(A_i)$ for all $i = 1, ..., n$, which in turn implies $\mu(A_i) = \nu(A_i)$ for all $i = 1, ..., n$.

Now, let $\{A_i\}_{i \in I}$ be any chain in Σ. Let Σ_J be the (finite) algebra generated by a finite subchain $\{A_i\}_{i \in J}$ and

$$\Lambda_J = \{\mu \in core(\nu) : \mu(A_j) = \nu(A_j) \text{ for all } j \in J\}.$$

Since $core(\nu)$ is weak*-compact, the set Λ_J is weak*-compact. Moreover, it is convex and, by what we just proved, $\Lambda_J \neq \varnothing$. It is easily seen that Λ_J is also extremal in $core(\nu)$.

The collection of weak*-compact sets $\{\Lambda_J\}_{\{J: J \subseteq I \text{ and } |J| < \infty\}}$ has the finite intersection property, and so its overall intersection $\bigcap_{\{J: J \subseteq I \text{ and } |J| < \infty\}} \Lambda_J$ is nonempty. Moreover, such intersection is extremal in $core(\nu)$. Being convex and weak*-compact, by the Krein-Milman Theorem $\bigcap_{\{J: J \subseteq I \text{ and } |J| < \infty\}} \Lambda_J$ has then an extreme point μ. We conclude that $\mu \in ext(core(\nu))$ and $\mu(A_i) = \nu(A_i)$ for all $i \in I$, as desired.

To complete the proof that (vi) implies (iv) it remains to show that $\nu \in bv(\Sigma)$. Since $core(\nu)$ is weak*-compact, it is bounded; i.e., there exists $M \in \mathbb{R}$ such that $\|\mu\| \leq M$ for all $\mu \in core(\nu)$. Since, given any finite chain $\varnothing = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = \Omega$, there exists $\mu \in core(\nu)$ such that $\mu(A_i) = \nu(A_i)$ for all $i = 0, ..., n$, we conclude that

$$\sum_{i=1}^{n} |\nu(A_i) - \nu(A_{i-1})| \leq \|\mu\| \leq M.$$

(iv) implies (v). Let $f \in B(\Sigma)$. Since ν_c is translation invariant, assume w.l.o.g. that $f \geq 0$. Consider the chain Γ of all upper sets $\{(f \geq t)\}_{t \in \mathbb{R}}$. Given any $\mu \in core(\nu)$, the following holds:

$$\nu_c(f) = \int \nu(f \geq t)\, dt \leq \int \mu(f \geq t)\, dt = \mu(f).$$

By (iv), there is $\mu \in core(\nu)$ such that $\nu(A) = \mu(A)$ for all $A \in \Gamma$. Hence,

$$\nu_c(f) = \int \nu(f \geq t)\, dt = \int \mu(f \geq t)\, dt = \mu(f),$$

and we conclude that $\nu_c(f) = \min_{\mu \in core(\nu)} \int f d\mu$.

Since $B(\Sigma)$ is supnorm dense in $\overline{B}(\Sigma)$, the supnorm continuous functional $\nu_c : \overline{B}(\Sigma) \to \mathbb{R}$ given by Lemma 36 is superlinear. By proceeding as before, we can show that:

$$\begin{aligned} core(\nu) &= \{\mu \in ba(\Sigma) : \mu_c(f) \geq \nu_c(f) \text{ for each } f \in B_0(\Sigma)\} \\ &= \left\{\mu \in ba(\Sigma) : \mu_c(f) \geq \nu_c(f) \text{ for each } f \in \overline{B}(\Sigma)\right\} \end{aligned}$$

Hence, by the Hahn-Banach Theorem:

$$\begin{aligned} \nu_c(f) &= \min\left\{\mu_c(f) : \mu \in ba(\Sigma) \text{ and } \mu_c(g) \geq \nu_c(g) \text{ for each } g \in \overline{B}(\Sigma)\right\} \\ &= \min\{\mu_c(f) : \mu \in ba(\Sigma) \text{ and } \mu_c(g) \geq \nu_c(g) \text{ for each } g \in B_0(\Sigma)\} \\ &= \min\{\mu_c(f) : \mu \in core(\nu)\} = \min_{\mu \in core(\nu)} \int f d\mu, \end{aligned}$$

as desired.

(v) implies (iii). Consider a finite chain $\{A_i\}_{i=1}^n$ and set $f = \sum_{i=1}^n 1_{A_i}$. By (v), there is $\mu \in core(\nu)$ such that $\mu(f) = \nu(f)$. By comonotonic additivity, $\sum_{i=1}^n \mu(A_i) = \sum_{i=1}^n \nu(A_i)$, and so $\sum_{i=1}^n [\mu(A_i) - \nu(A_i)] = 0$. Since $\mu \geq \nu$ we conclude that $\mu(A_i) = \nu(A_i)$ for each $i = 1, ..., n$, as desired.

As (iii) trivially implies (ii), it remains to show that (ii) implies (i). Given any A and B, by (ii) there is $\mu \in core(\nu)$ such that $\nu(A) = \mu(A)$ and $\nu(B) = \mu(B)$. Hence,

$$\begin{aligned} \nu(A \cap B) + \nu(A \cup B) &= \mu(A \cap B) + \mu(A \cup B) \\ &= \mu(B) + \mu(A) \geq \nu(B) + \nu(A), \end{aligned}$$

where the last inequality follows from $\mu \in core(\nu)$. ■

We close with some characterizations of convexity through properties of subgames, thus providing an "hereditary" perspective on it.

Theorem 39 *For a bounded game ν, the following conditions are equivalent:*

(i) ν is convex,

(ii) each subgame of ν is exact,

(iii) ν is totally balanced and, given any $A \subseteq B$, each charge in $core(\nu_A)$ has an extension belonging to $core(\nu_B)$.

(iv) ν is balanced and, given any finite subalgebra Σ_0 of Σ, each charge in core (ν_{Σ_0}) has an extension on Σ belonging to core (ν).

The equivalence between (i) and (iv) is essentially due to Kelley [35] (see also [12, p. 218]), that between (i) and (ii) to Biswas et al. [6, p. 10], and that between (i) and (iii) to Einy and Shitovitz [21, pp. 197-199].

The second part of condition (iii) is a property introduced by Kikuta and Shapley [37]. They call *extendable* the games satisfying this property for $B = \Omega$, which turns out to be useful in studying the von Neumann-Morgenstern stability of cores.[11]

The proof of Theorem 39 uses the following straightforward lemma. In this regard, notice that Schmeidler [57, p. 219] gives an example of an exact game with four players that is not convex.

Lemma 40 *A finite game with at most three players is exact if and only if it is convex.*

Proof of Theorem 39. For convenience, we first prove the equivalence between (i)-(iii), and then that between (i) and (iv).

(ii) implies (i). Given any A and B, consider the subgame $\nu_{A\cup B}$. By (ii), there is $\mu \in core\,(\nu_{A\cup B})$ such that $\mu\,(A\cap B) = \nu_{A\cup B}\,(A\cap B)$. Hence,

$$\begin{aligned}\nu\,(A\cup B)+\nu\,(A\cap B) &= \nu_{A\cup B}\,(A\cup B)+\nu_{A\cup B}\,(A\cap B)\\ &= \mu\,(A\cup B)+\mu\,(A\cap B)=\mu\,(A)+\mu\,(B)\\ &\geq \nu_{A\cup B}\,(A)+\nu_{A\cup B}\,(B)=\nu\,(A)+\nu\,(B)\,,\end{aligned}$$

as desired.

(i) implies (iii). Since $A \subseteq B$, the space $B_0\,(\Sigma_A)$ of simple Σ_A-measurable functions can be regarded as a vector subspace of $B_0\,(\Sigma_B)$. Let $\mu \in core\,(\nu_A)$. Given any $f \in B_0^+\,(\Sigma_A)$, it holds $\nu_{A,c}\,(f) = \nu_{B,c}\,(f)$, where $\nu_{A,c} : B_0\,(\Sigma_A) \to \mathbb{R}$ is the Choquet functional induced by the subgame ν_A ($\nu_{B,c}$ is similarly defined). Therefore, $\mu\,(f) \geq \nu_{B,c}\,(f)$ for all $f \in B_0^+\,(\Sigma_A)$.

Given any $f \in B_0\,(\Sigma_A)$, there is $k > 0$ large enough so that $f + k1_A \in B_0^+\,(\Sigma_A)$. Since $\mu\,(A) = \nu\,(A)$, by Theorem 35 we have:

$$\mu\,(f)+k\mu\,(A)=\mu\,(f+k1_A)\geq\nu_{B,c}\,(f+k1_A)\geq\nu_{B,c}\,(f)+k\nu\,(A)=\nu_{B,c}\,(f)+k\mu\,(A)\,.$$

[11] See, e.g., [6] and the references therein contained. For characterizations of convexity and exactness related to stability, see [36] and [64].

Hence, $\mu(f) \geq \nu_{B,c}(f)$. We conclude that $\mu(f) \geq \nu_{B,c}(f)$ for all $f \in B_0(\Sigma_A)$.

By the Hahn-Banach Theorem, there exists a charge $\mu^* : \Sigma_B \to \mathbb{R}$ which extends μ and such that $\mu^*(f) \geq \nu_{B,c}(f)$ for all $f \in B_0(\Sigma_B)$. Hence, $\mu^* \in core(\nu_B)$.

(iii) implies (ii). Given any B, let ν_B be the associated subgame. Given any $A \subseteq B$, let $\mu \in core(\nu_A)$. By hypothesis, there is $\mu^* \in core(\nu_B)$ that extends μ. Hence, $\mu^*(A) = \mu(A) = \nu_A(A) = \nu_B(A)$, which implies that ν_B is exact, as desired.

To complete the proof it remains to show that (iv) is equivalent to (i).

(i) implies (iv). Let $\mu \in core(\nu_{\Sigma_0})$. By Theorem 38, $\nu \in bv(\Sigma)$, and so $\nu_c : B_0(\Sigma) \to \mathbb{R}$ is superlinear by Theorem 35. By the Hahn-Banach Theorem, there is $\mu^* \in ba(\Sigma)$ that extends μ and such that $\mu^*(f) \geq \nu_c(f)$ for all $f \in B_0(\Sigma)$. Hence, $\mu^* \in core(\nu)$.

(iv) implies (i). If $\mu \in core(\nu)$, then its restriction μ_{Σ_0} on any subalgebra Σ_0 belongs to $core(\nu_{\Sigma_0})$. Therefore, the fact that ν is balanced implies that $core(\nu_{\Sigma_0}) \neq \varnothing$ for each Σ_0. In particular, (iv) implies that:

$$core(\nu_{\Sigma_0}) = \left\{\mu_{\Sigma_0} \in ba(\Sigma_0) : \mu \in core(\nu)\right\}. \tag{20}$$

Given any A, consider first the finite subalgebra $\Sigma_0 = \{\varnothing, A, A^c, \Omega\}$. It is easy to see that there exists an element of $core(\nu_{\Sigma_0})$ that on A takes on value $\nu(A)$. By (20), this amounts to say that there exists $\mu \in core(\nu)$ such that $\mu(A) = \mu_{\Sigma_0}(A) = \nu(A)$. We conclude that ν is exact.

Given any A and B, consider the finite subalgebra Σ_0 generated by the partition $\{A \cap B, (A \cup B)^c, A \cup B \backslash A \cap B\}$. Let $C \in \Sigma_0$. As ν is exact, there is $\mu \in core(\nu)$ such that $\mu(C) = \nu(C)$. As $\mu_{\Sigma_0} \in core(\nu_{\Sigma_0})$, we have $\mu_{\Sigma_0}(C) = \nu_{\Sigma_0}(C)$ and so ν_{Σ_0} is exact. Since Σ_0 is generated by a partition consisting of three elements, Lemma 40 then implies that ν_{Σ_0} is convex. Since $A \cap B, A \cup B \in \Sigma_0$, by Theorem 38 and (20) there is $\mu \in core(\nu)$ such that $\mu_{\Sigma_0}(A \cap B) = \nu_{\Sigma_0}(A \cap B)$ and $\mu_{\Sigma_0}(A \cup B) = \nu_{\Sigma_0}(A \cup B)$. Hence,

$$\begin{aligned}
\nu(A \cup B) + \nu(A \cap B) &= \nu_{\Sigma_0}(A \cup B) + \nu_{\Sigma_0}(A \cap B) \\
&= \mu_{\Sigma_0}(A \cup B) + \mu_{\Sigma_0}(A \cap B) = \mu(A \cup B) + \mu(A \cap B) \\
&= \mu(A) + \mu(B) \geq \nu(A) + \nu(B),
\end{aligned}$$

which shows that ν is convex. ■

6 Finite Games

6.1 The Space of Finite Games

Games defined on finite spaces have some noteworthy peculiar properties, thanks to the special form of their domain. We devote this section to their study.[12]

Let Ω be a finite set $\{\omega_1, ..., \omega_n\}$ of n players and Σ its power set. V_n denotes the space of all finite games on Σ, which is a vector space under the setwise operations $\nu_1 + \nu_2$ and $\alpha\nu$ for $\nu_1, \nu_2, \nu \in V_n$ and $\alpha \in \mathbb{R}$.

The next result, due to Shapley [61, Lemma 3], shows the crucial importance of unanimity games, introduced in Example 20.

Theorem 41 *Unanimity games form a basis for the* $\left(2^{|\Sigma|} - 1\right)$*-dimensional vector space* V_n*. For any* $\nu \in V_n$*, the unique coefficients satisfying* $\nu = \sum_{\varnothing \neq A \in \Sigma} \alpha_A^\nu u_A$ *are given by*

$$\alpha_A^\nu = \sum_{B \subseteq A} (-1)^{|A|-|B|} \nu(B). \tag{21}$$

Proof. We first show that unanimity games are linearly independent in V_n. Suppose $\sum_{i=1}^n \alpha_i u_{A_i} = \theta$, where θ is the trivial game such that $\theta(A) = 0$ for each A. We want to show that $\alpha_i = 0$ for each $i = 1, ..., n$. Suppose, *per contra*, that there is a subset $I \subseteq \{1, ..., n\}$ such that $\alpha_i \neq 0$ for each $i \in I$. As in [45, p. 440], let $i_0 \in I$ be such that A_{i_0} is of minimal size among the coalitions $\{A_i\}_{i \in I}$. By construction, $\alpha_i = 0$ for each i such that $A_i \subsetneqq A_{i_0}$, so that

$$0 = \sum_{i=1}^n \alpha_i u_{A_i}(A_{i_0}) = \sum_{\{i: A_i \subseteq A_{i_0}\}} \alpha_i u_{A_i}(A_{i_0}) = \alpha_{i_0},$$

a contradiction.

To complete the proof it remains to prove that, for each $\nu \in V_n$, it holds:

$$\nu = \sum_{\varnothing \neq A \in \Sigma} \left(\sum_{B \subseteq A} (-1)^{|A|-|B|} \nu(B) \right) u_A.$$

The needed combinatorial argument is detailed in, e.g., [48, p. 263], to which we refer the reader. ■

[12] Needless to say, the properties we will establish for finite games also hold for games defined on finite algebras of subsets of infinite spaces.

Example 42 For a charge μ we have:

$$\alpha_A^\mu = \begin{cases} \mu(\omega) & \text{if } A = \{\omega\} \\ 0 & \text{else} \end{cases}$$

That is, a game is additive if and only if all coefficients in (21) associated with non singletons are zero. ▲

Example 43 Let $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$. Its first difference is $\Delta f(n) = f(n+1) - f(n)$. By iteration, the k-order difference is $\Delta^k f = \Delta\Delta^{k-1} f$. Consider the scalar measure game $\nu : 2^{\{1,...,n\}} \to \mathbb{R}$ defined by $\nu(A) = f(|A|)$ for each $A \subseteq \{1, ..., n\}$. The following holds:

$$\alpha_A^\nu = \Delta^{|A|} f(0). \tag{22}$$

To see why this the case, observe that (21) implies

$$\alpha_A^\nu = f(m) - \binom{m}{1} f(m-1) + \binom{m}{2} f(m-2) - .... = \\ = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(m-k),$$

where we set $|A| = m$. Denote by $I : \mathbb{N} \to \mathbb{N}$ the identity operator and by $S : \mathbb{N} \to \mathbb{N}$ the shift operator defined by $Sf(m) = f(m+1)$. As $\Delta = S - I$, we have

$$\Delta^m = (S - I)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} S^{m-k}.$$

Hence

$$\Delta^m f(0) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} S^{m-k} f(0) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(m-k),$$

and so (22) holds. ▲

By Theorem 41, each game ν is uniquely determined by the coefficients $\{\alpha_A^\nu\}$ given by (21). A natural question is whether there is a significant class of games identified by the requirement that all such coefficients be positive. Fortunately, Theorem 46 will show that there is such a class, which we now introduce.

A game $\nu : \Sigma \to \mathbb{R}$ is

14. *monotone of order* k (with $k \geq 2$) if, for every $A_1, ..., A_k \in \Sigma$,

$$\nu\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\{I:\varnothing\neq I\subseteq\{1,...,k\}\}} (-1)^{|I|+1} \nu\left(\bigcap_{i\in I} A_i\right). \tag{23}$$

15. *totally monotone* if it is positive and k-monotone for all $k \geq 2$.

16. a *belief function* if it is a totally monotone probability.

These definitions work for any algebra Σ, not necessarily finite. For $k = 2$, we get back to convexity. Hence, totally monotone games are convex, though the converse is false. When ν is a charge, in (23) we have an equality.

Totally monotone games are studied at length in Choquet (1953), and belief functions play a central role in the works of Dempster and Shafer (see [15], [16], and [60]). They are also related to the theory of Mobius transforms pioneered by Rota [51], as detailed in [10] and [28] (see also Subsection 4.4 below).

Example 44 All $\{0, 1\}$-valued convex games (e.g., unanimity games) are totally monotone (see, e.g., [40, p. 1005] for a proof). ▲

Example 45 Let $(\Omega_1, \Sigma_1, P_1)$ be a probability space and Ω_2 a finite space. A correspondence $f : \Omega_1 \to 2^{\Omega_2}$ is a random set if it is measurable, that is, $f^{-1}(A) = \{\omega \in \Omega_1 : f(\omega) \subseteq A\} \in \Sigma_1$ for each $A \subseteq \Omega_2$. Consider the distribution $\nu_f : \Sigma_2 \to \mathbb{R}$ induced by a random set f, defined by $\nu_f(A) = P(f^{-1}(A))$ for each $A \in \Sigma_2$. The distribution ν_f is a belief function (see, e.g., [46]). Random sets reduce to standard random variables when the images $f(\omega)$ are singletons; in this case, ν_f is the usual additive distribution induced by a random variable f. Under suitable topological conditions, random sets with values in infinite spaces Ω_2 can also be considered (there is a large literature on them; see, e.g., [54]). ▲

We can now state the announced result.

Theorem 46 *Let ν be a game defined on the power set Σ of a finite space Ω. Then, the coefficients given by (21) are all positive if and only if ν is totally monotone.*

Remark. This theorem is essentially due to Dempster and Shafer (see [60]). It has been extended to games on lattices by Gilboa and Lehrer [25].

Proof. "If part". Suppose ν is totally monotone. If $|A| = 1$, then $\alpha_A^\nu = \nu(A) \geq 0$. Suppose $|A| > 1$ and set $A = \{\omega_1, ..., \omega_k\}$ and $A_i = A \setminus \{\omega_i\}$ for each $i = 1, ..., k$. We have

$$\begin{aligned} \alpha_A^\nu &= \sum_{B \subseteq A} (-1)^{|A|-|B|} \nu(B) \\ &= \nu(A) - \sum_i \nu(A_i) + \sum_{i \neq j} \nu(A_i \cap A_j) - .. + (-1)^k \nu(A_1 \cap ... \cap A_k) \\ &= \nu(A) - \sum_{\varnothing \neq I \subseteq \{1,..,k\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right). \end{aligned}$$

As $A = \bigcup_{i=1}^k A_i$, we then have

$$\alpha_A^\nu = \nu\left(\bigcup_{i=1}^k A_i\right) - \sum_{\varnothing \neq I \subseteq \{1,..,k\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right),$$

so that $\alpha_A^\nu \geq 0$, as desired.

"If" part. Suppose $\alpha_A^\nu \geq 0$ for each $\varnothing \neq A \in \Sigma$. By Example 44, each unanimity game u_A is totally monotone. Hence, by Theorem 41, we have $\nu = \sum_{\varnothing \neq A} \alpha_A^\nu u_A$. The positive linear combination of totally monotone game is clearly totally monotone. We infer that ν is totally monotone as desired. ■

Example 47 Given a function $f : \mathbb{N} \to \mathbb{R}$ with $f(0) = 0$, each scalar measure game $f(|A|)$ is totally monotone if and only if f is absolutely monotone à la Bernstein, that is, $\Delta^k f(n) \geq 0$ for each n and k (see [69]). By (22) and by Theorem 46, to prove this fact it is enough to show that f is absolutely monotone if and only if $\Delta^k f(0) \geq 0$ for each k. As

$$\Delta^n S^k = \Delta^n [\Delta + I]^k = \Delta^n \sum_{r=1}^k \binom{k}{r} \Delta^r = \sum_{r=1}^k \binom{k}{r} \Delta^{r+n},$$

we get

$$\Delta^n f(k) = \sum_{r=1}^k \binom{k}{r} \Delta^{r+n} f(0) \geq 0,$$

which gives the desired conclusion. ▲

Totally monotone games are therefore the convex cone of V_n consisting of all its elements featuring positive coefficients in (21). Denote this cone by V_n^+; being pointed,[13] it induces a partial order $\succeq$ on V_n defined by $\nu \succeq \nu'$ if $\nu - \nu' \in V_n^+$. In particular, $\nu \succeq \theta$ if and only if ν is totally monotone, so that $V_n^+ = \{\nu \in V_n : \nu \succeq \theta\}$.

The partial order $\succeq$ makes V_n an ordered vector space. More is true, under the lattice operations $\vee$ and $\wedge$ induced by $\succeq$.[14]

Lemma 48 *The ordered vector space $(V_n, \succeq)$ is a Riesz space with lattice operations given by*

$$\nu_1 \vee \nu_2 = \Sigma_{\varnothing \neq A \in \Sigma} \left(\alpha_A^{\nu_1} \vee \alpha_A^{\nu_2}\right) u_A,$$

and

$$\nu_1 \wedge \nu_2 = \Sigma_{\varnothing \neq A \in \Sigma} \left(\alpha_A^{\nu_1} \wedge \alpha_A^{\nu_2}\right) u_A,$$

for each ν_1 and ν_2 in V_n.

Proof. We only prove the result for $\nu_1 \vee \nu_2$, as a similar argument can be used for $\nu_1 \wedge \nu_2$. Set $\widehat{\nu} = \Sigma_{\varnothing \neq A \in \Sigma} \left(\alpha_A^{\nu_1} \vee \alpha_A^{\nu_2}\right) u_A$. We want to show that $\widehat{\nu} = \nu_1 \vee \nu_2$. First observe that, for $i = 1, 2$,

$$\widehat{\nu} - \nu_i = \Sigma_{\varnothing \neq A \in \Sigma} \left[\left(\alpha_A^{\nu_1} \vee \alpha_A^{\nu_2}\right) - \alpha_A^{\nu_i}\right] u_A.$$

Hence, by Theorem 46, $\widehat{\nu} - \nu_i \in V_n^+$, all coefficients $\left[\left(\alpha_A^{\nu_1} \vee \alpha_A^{\nu_2}\right) - \alpha_A^{\nu_i}\right]$ being positive. This shows that $\widehat{\nu}$ is an upper bound for $\{\nu_1, \nu_2\}$. It remains to show that it is the least such bound, i.e., $\widehat{\nu}' \succeq \widehat{\nu}$ for any game $\widehat{\nu}'$ such that $\widehat{\nu}' \succeq \nu_i$ for $i = 1, 2$.

As $\widehat{\nu}' - \nu_i \in V_n^+$, it holds

$$\Sigma_{\varnothing \neq A \in \Sigma} \alpha_A^{\widehat{\nu}'} u_A - \Sigma_{\varnothing \neq A \in \Sigma} \alpha_A^{\nu_i} u_A = \Sigma_{\varnothing \neq A \in \Sigma} \alpha_A^{\widehat{\nu}' - \nu_i} u_A \in V_n^+.$$

By Theorem 46, $\alpha_A^{\widehat{\nu}' - \nu_i} \geq 0$ for each A, and so $\alpha_A^{\widehat{\nu}'} \geq \alpha_A^{\nu_i}$ for each A. Therefore, $\alpha_A^{\widehat{\nu}'} \geq \alpha_A^{\nu_1} \vee \alpha_A^{\nu_2}$ for each A, and so the difference

$$\widehat{\nu}' - \widehat{\nu} = \Sigma_{\varnothing \neq A \in \Sigma} \left[\alpha_A^{\widehat{\nu}'} - \left(\alpha_A^{\nu_1} \vee \alpha_A^{\nu_2}\right)\right] u_A$$

[13] I.e., $V^+ \cap (-V^+) = \{0\}$.

[14] See [1, pp. 263-330] for a definition of these lattice operations, as well as for all notions on vector lattices needed in the sequel.

belongs to V_n^+ by Theorem 46, all the coefficients $\alpha_A^{\widehat{\nu}'} - (\alpha_A^{\nu_1} \vee \alpha_A^{\nu_2})$ being positive. We conclude that $\widehat{\nu}' \succeq \nu_i$ for each i, as desired. ■

The Riesz space $(V_n, \succeq)$ is lattice isomorphic to the Euclidean space $\left(\mathbb{R}^{2^{|\Sigma|}-1}, \geq\right)$.

Lemma 49 *The function $T : V_n \to \mathbb{R}^{2^{|\Sigma|}-1}$ defined by*

$$T(\nu) = (\alpha_A^\nu) \qquad \text{for all } \nu \in V_n$$

is a lattice preserving isomorphism between $(V_n, \succeq)$ and $\left(\mathbb{R}^{2^{|\Sigma|}-1}, \geq\right)$.

Proof. By Theorem 41, the vector (α_A^ν) is uniquely determined. Hence, T is one-to-one. Now, let $\nu_1, \nu_2 \in V_n$ and $\alpha, \beta \in \mathbb{R}$. By Theorem 41,

$$\begin{aligned} T(\alpha\nu_1 + \beta\nu_2) &= \left(\alpha_A^{\alpha\nu_1+\beta\nu_2}\right) = \left(\sum_{B \subseteq A} (-1)^{|A|-|B|} (\alpha\nu_1 + \beta\nu_2)(B)\right) \\ &= \left(\alpha \sum_{B \subseteq A} (-1)^{|A|-|B|} \nu_1(B) + \beta \sum_{B \subseteq A} (-1)^{|A|-|B|} \nu_2(B)\right) \\ &= \alpha T(\nu_1) + \beta T(\nu_2), \end{aligned}$$

and so T is an isomorphism. Moreover, by Lemma 48,

$$\begin{aligned} T(\nu_1 \vee \nu_2) &= T(\nu_1) \vee T(\nu_2) = (\alpha_A^{\nu_1} \vee \alpha_A^{\nu_2}) \in \mathbb{R}^{2^{|\Sigma|}-1}, \\ T(\nu_1 \wedge \nu_2) &= T(\nu_1) \wedge T(\nu_2) = (\alpha_A^{\nu_1} \wedge \alpha_A^{\nu_2}) \in \mathbb{R}^{2^{|\Sigma|}-1}, \end{aligned}$$

as desired. ■

By Lemma 48, the positive ν^+ and negative ν^- parts of a game ν, defined by $\nu^+ = \nu \vee 0$ and $\nu^- = -(\nu \wedge 0)$, are given by:

$$\begin{aligned} \nu^+ &= \Sigma_{\varnothing \neq A \in \Sigma} (\alpha_A^\nu \vee 0)\, u_A, \\ \nu^- &= \Sigma_{\varnothing \neq A \in \Sigma} (\alpha_A^\nu \wedge 0)\, u_A. \end{aligned}$$

The absolute value $|\nu|$, defined by $|\nu| = \nu^+ + \nu^-$, is then given by

$$|\nu| = \Sigma_{\varnothing \neq A \in \Sigma} |\alpha_A^\nu|\, u_A.$$

Notice that

$$\nu^+ = T^{-1}\left((\alpha_A^\nu)^+\right),\ \nu^- = T^{-1}\left((\alpha_A^\nu)^-\right),\ \text{and}\ |\nu| = T^{-1}\left(|\alpha_A^\nu|\right),$$

in accordance with Lemma 49.

The associated norm $\|\cdot\|_c$ is given by $\|\nu\|_c = |\nu|(\Omega) = \nu^+(\Omega) + \nu^-(\Omega)$ for each $\nu \in V_n$, that is,

$$\|\nu\|_c = \Sigma_{\varnothing \neq A \in \Sigma} |\alpha_A^\nu| = \|T(\nu)\|_1. \tag{24}$$

Following Gilboa and Schmeidler [27], we call $\|\cdot\|_c$ the *composition norm.* It is an L-norm since $\|\nu_1 + \nu_2\|_c = \|\nu_1\|_c + \|\nu_1\|_c$ whenever ν_1 and ν_2 belong to V_n^+. As a result, $(V_n, \succeq, \|\cdot\|_c)$ is an AL-space.

Since

$$\|\nu\|_c = \Sigma_{\varnothing \neq A \in \Sigma} |\alpha_A^\nu| = \|T(\nu)\|_1, \tag{25}$$

where $\|\cdot\|_1$ is the l_1-norm of $\mathbb{R}^{2^{|\Sigma|}-1}$, the isomorphism T is therefore an isometry between $(V_n, \|\cdot\|_c)$ and $\left(\mathbb{R}^{2^{|\Sigma|}-1}, \|\cdot\|_1\right)$.[15]

Summing up:

Theorem 50 *There is a lattice preserving and isometric isomorphism T between the AL-spaces $(V_n, \succeq, \|\cdot\|_c)$ and $\left(\mathbb{R}^{2^{|\Sigma|}-1}, \geq, \|\cdot\|_1\right)$ determined by the identity*

$$\nu = \Sigma_{\varnothing \neq A \in \Sigma} \alpha_A u_A. \tag{26}$$

Moreover, ν is totally monotone if and only if the corresponding vector (α_A) in $\mathbb{R}^{2^{|\Sigma|}-1}$ is nonnegative.

In other words, for each ν in V_n there is a unique (α_A) in $\mathbb{R}^{2^{|\Sigma|}-1}$ such that (26) holds; conversely, for each vector (α_A) in $\mathbb{R}^{2^{|\Sigma|}-1}$ there is a unique ν in V_n such that (26) holds. Moreover, the correspondence T between ν and (α_A) is linear, lattice preserving, and isometric.

Consider the restriction of the partial order $\succeq$ on $ba(\Sigma)$, the vector subspace of V_n consisting of charges. Since $ba^+(\Sigma) = V_n^+ \cap ba(\Sigma)$, given any μ_1 and μ_2 in $ba(\Sigma)$, we have $\mu_1 \succeq \mu_2$ if and only if $\mu_1 - \mu_2 \in ba^+(\Sigma)$. Equivalently, $\mu_1 \succeq \mu_2$ if and only if $\mu_1 \geq \mu_2$ setwise, that is, $\mu_1(A) \geq \mu_2(A)$ for

[15] The l_1-norm $\|\cdot\|_1$ of $\mathbb{R}^n$ is given by $\|x\|_1 = \sum_{i=1}^n |x_i|$ for each $x \in \mathbb{R}^n$.

each A. This is the standard partial order studied on $ba(\Sigma)$ (see, e.g., [5]), which can therefore be viewed as the restriction of $\succeq$ on $ba(\Sigma)$. As a result, the standard lattice structure on $ba(\Sigma)$ coincides with the one it inherits as a subspace of $(V_n, \succeq)$. In particular, on $ba(\Sigma)$ the norm $\|\cdot\|_c$ reduces to the total variation norm $\|\cdot\|$.

All this shows that the standard structures on $ba(\Sigma)$ studied in measure theory are consistent with the ones we have identified on V_n so far. In the sequel we will denote by $\succeq_{ba}$ the restriction of $\succeq$ on $ba(\Sigma)$.

6.2 A Decomposition

The lattice structure of V_n suggests the possibility of achieving a decomposition à la Riesz for finite games. Given the close connection between $\|\cdot\|$ and the l_1-norm $\|\cdot\|_1$ established in Theorem 50, it is natural to expect that such decomposition would resemble the one available for the familiar l_1-norm. For this reason, we first recall a simple decomposition result for the l_1-norm.

Lemma 51 *Given any $z \in \mathbb{R}^n$, the vectors z^+ and z^- are the unique vectors in $\mathbb{R}^n_+$ such that*

$$z = z^+ - z^-, \tag{27}$$

and

$$\|z\|_1 = \left\|z^+\right\|_1 + \left\|z^-\right\|_1. \tag{28}$$

Proof. Clearly, the decomposition $z = z^+ - z^-$ satisfies (28). Suppose $x, y \in \mathbb{R}^n_+$ satisfy (27). We want to show that $x \geq z^+$ and $y \geq z^-$. As $x = z + y$, $x \geq z$, we have $x \geq z^+$. Likewise, $y = x - z \Longrightarrow y \geq -z \Longrightarrow y \geq z^-$.

On the other hand, we have

$$\|x\|_1 + \|y\|_1 \geq \left\|z^+\right\|_1 + \left\|z^-\right\|_1 = \|z\|_1.$$

As $\|x\|_1 \geq \|z^+\|_1$ and $\|y\|_1 \geq \|z^-\|_1$, to get (28) we must have $\|x\|_1 = \|z^+\|_1$ and $\|y\|_1 = \|z^-\|_1$. Hence, $x = z^+$ and $y = z^-$. ■

Lemma 51 leads to the following decomposition result, which generalizes in our finite setting the Jordan Decomposition Theorem for charges. Versions of this result for finite and infinite games have been proved by Revuz [50], Gilboa and Schmeidler [26] and [27], and Marinacci [40].

Theorem 52 *Given any $\nu \in V_n$, the games ν^+ and ν^- are the unique totally monotone games such that*

$$\nu = \nu^+ - \nu^- \tag{29}$$

and

$$\|\nu\|_c = \left\|\nu^+\right\|_c + \left\|\nu^-\right\|_c. \tag{30}$$

Proof. Let ν_1 and ν_2 be any two games in V_n^+ satisfying (29) and (30). Then, the positive vectors $T(\nu_1)$ and $T(\nu_2)$ of $\mathbb{R}_+^{2^{|\Sigma|}-1}$ are such that

$$\begin{aligned} T(\nu) &= T(\nu_1) - T(\nu_2), \text{ and} \\ \|T(\nu)\|_1 &= \|T(\nu_1)\|_1 + \|T(\nu_2)\|_1. \end{aligned}$$

By Lemma 51, $T(\nu_1) = T(\nu)^+ = T(\nu^+)$ and $T(\nu_1) = T(\nu)^- = T(\nu^-)$. Since T is an isomorphism, we conclude that $\nu_1 = \nu^+$ and $\nu_2 = \nu^-$, as desired. ■

6.3 Additive Representation

By Theorem 41, each finite game ν can be uniquely written as

$$\nu = \sum_{\varnothing \neq A \in \Sigma} \alpha_A^\nu u_A, \tag{31}$$

Let Σ' be the collection of all nonempty sets of Σ, that is, $\Sigma' = \{A \in \Sigma : A \neq \varnothing\}$. The collection Σ' can be viewed as new space, whose "points" are the nonempty sets of Σ. By identifying Ω with the collection of all singletons $\{\{\omega\} : \omega \in \Omega\}$, we can actually view the space Σ' as an enlargement of the original space Ω.

Define on the power set $2^{\Sigma'}$ of the space Σ' a charge μ_ν as follows: $\mu_\nu(A) = \alpha_A^\nu$ for each $A \in \Sigma'$. By additivity, this is enough to define the charge μ_ν on the entire power set $2^{\Sigma'}$. For example, for the set $\{A, B\} \in 2^{\Sigma'}$ we have $\mu_\nu(\{A, B\}) = \alpha_A^\nu + \alpha_B^\nu$; more generally, given any collection $\{A_1, ..., A_n\} \in 2^{\Sigma'}$, we have $\mu_\nu(\{A_1, ..., A_n\}) = \sum_{i=1}^n \alpha_{A_i}^\nu$.

Each game ν is thus associated with a charge μ_ν on $2^{\Sigma'}$. Denote by $I : V_n \to ba\left(2^{\Sigma'}\right)$ this correspondence $\nu \mapsto \mu_\nu$, which is well defined by Theorem 41. It is also linear, that is, $I(\alpha\nu_1 + \beta\nu_2) = \alpha I(\nu_1) + \beta I(\nu_2)$ for all $\alpha, \beta \in \mathbb{R}$ and all $\nu_1, \nu_2 \in V_n$.

The linear correspondence I provides some noteworthy insights into Choquet integrals. To see why, given a set E consider the function $\widetilde{1}_E : \Sigma' \to \mathbb{R}$ defined by

$$\widetilde{1}_E(A) = \int_\Omega 1_E du_A = u_A(E) = \begin{cases} 1 & A \subseteq E \\ 0 & \text{else} \end{cases}$$

for each A. If we set $\widetilde{E} = \{A \in \Sigma' : A \subseteq E\}$, then $\widetilde{1}_E = 1_{\widetilde{E}}$. That is, $\widetilde{1}_E$ is a characteristic function on the enlarged space Σ'.

Using μ_ν and $\widetilde{1}_E$, we can rewrite (31) as

$$\nu(E) = \sum_{A \in \Sigma'} \widetilde{1}_E(A)\,\mu_\nu(A) = \int_{\Sigma'} \widetilde{1}_E d\mu_\nu = \int_{\Sigma'} 1_{\widetilde{E}} d\mu_\nu = \mu_\nu\left(\widetilde{E}\right)$$

for each $E \in \Sigma$. Equivalently,

$$\int_\Omega 1_E d\nu = \int_{\Sigma'} 1_{\widetilde{E}} d\mu_\nu \qquad \text{for each } E \in \Sigma. \tag{32}$$

Therefore, thanks to the linear correspondence I we can represent the Choquet integral $\int 1_E d\nu$ as a standard additive integral on the enlarged space Σ'. In this extended domain, the set E of Σ is replaced by the set $\widetilde{E} = \{A \in \Sigma' : A \subseteq E\}$ of Σ'. We call $\int_{\Sigma'} 1_{\widetilde{E}} d\mu_\nu$ *the additive representation* of $\int 1_E d\nu$.

In a sense, (32) says that the Choquet integral $\int 1_E d\nu$ can be viewed as a "zipped" version of the additive integral $\int_{\Sigma'} 1_{\widetilde{E}} d\mu_\nu$. The trade-off here is between a more economical domain – i.e., (Ω, Σ) rather than $\left(\Sigma', 2^{\Sigma'}\right)$ – and a better behaved integral – i.e., the additive integral rather than the non-additive one.

In any case, to compute both representations we need to know the $2^{|\Omega|-1}$ values of ν and μ_ν, respectively; hence, both representations involve the same amount of information, though processed in different ways.

Next we formally collect the relevant properties of the additive representation. Observe that the correspondence I is actually an isomorphism.[16]

Theorem 53 *There is a lattice preserving and isometric isomorphism I between the AL-spaces $(V_n, \succeq, \|\cdot\|)$ and $\left(ba\left(2^{\Sigma'}\right), \succeq_{ba}, \|\cdot\|\right)$ determined by the identity*

$$\nu(E) = \mu\left(\widetilde{E}\right) \qquad \textit{for each } E \in \Sigma. \tag{33}$$

[16] In the statement $\succeq_{ba}$ denotes the restriction of $\succeq$ on $ba(\Sigma)$ as discussed right after Corollary 50.

The game ν is totally monotone if and only if the corresponding μ is non-negative.

Versions of this result for finite and infinite games can be found in Revuz [50], Gilboa and Schmeidler [26] and [27], Marinacci [40], and Philippe, Debs, and Jaffray [49]. Denneberg [18] provides an overview and alternative proofs of some of these results.

Proof. Given a charge $\mu \in ba\left(2^{\Sigma'}\right)$, the set function ν on Σ defined by (33) is clearly a game. As to the converse, the charge μ_ν defined above belongs to $ba\left(2^{\Sigma'}\right)$ and satisfies (33). It is also the unique charge in $ba\left(2^{\Sigma'}\right)$ satisfying (33). In fact, let μ be any other charge in $ba\left(2^{\Sigma'}\right)$ satisfying (33). Consider the collection $\widetilde{\Sigma} = \left\{\widetilde{E} : E \in \Sigma\right\}$ of subsets of Σ'. As

$$\widetilde{E_1 \cap E_2} = \widetilde{E}_1 \cap \widetilde{E}_2 \text{ and } \widetilde{E_1 \cup E_2} \supseteq \widetilde{E}_1 \cup \widetilde{E}_2,$$

the collection $\widetilde{\Sigma}$ is, in general, only closed under intersections, that is, it is a π-class (see, e.g., [1, p. 132]). As μ and μ_ν coincide on a π-class, they coincide on the algebra $\mathcal{A}\left(\widetilde{\Sigma}\right)$ generated by $\widetilde{\Sigma}$ (see, e.g., [1, Thm 9.10]). But, $\mathcal{A}\left(\widetilde{\Sigma}\right)$ coincides with the power set $2^{\Sigma'}$ of Σ'. For, $\mathcal{A}\left(\widetilde{\Sigma}\right)$ contains all singletons: given $A \in \Sigma'$, we have $\{A\} = \widetilde{A} \backslash \widetilde{A - \omega}$ for any $\omega \in A$. As a result, μ and μ_ν coincide on the power set $2^{\Sigma'}$, thus proving that μ_ν is the unique charge in $ba\left(2^{\Sigma'}\right)$ satisfying (33).

All this shows that the linear correspondence I we introduced above is an isomorphism between V_n and $ba\left(2^{\Sigma'}\right)$. It is also an isometry: the equality $\|I(\nu)\| = \|\nu\|_r$ follows from

$$\|\mu_\nu\| = \sum_{A \in \Sigma'} |\mu_\nu(A)| = \sum_{A \in \Sigma'} |\alpha_A^\nu| = \|\nu\|_r .$$

It remains to show that I is lattice preserving. We will only consider $\vee$, the argument for $\wedge$ being similar. For each $A \in \Sigma'$, we have:

$$\mu_{\nu_1 \vee \nu_2}(A) = \alpha_A^{\nu_1} \vee \alpha_A^{\nu_2} = \max\left\{\mu_{\nu_1}(A), \mu_{\nu_2}(A)\right\} = \left(\mu_{\nu_1} \vee \mu_{\nu_2}\right)(A),$$

as desired (the last equality holds because A is a singleton when viewed as a member of Σ'). ■

The additive representation is not limited to integrals of characteristic functions, but it holds for all functions in $B(\Sigma)$. To see why this is the case, observe that the additivity of the Riemann integral immediately implies that the Choquet integral is linear on games, that is, $\int fd(\nu_1+\nu_2) = \int fd\nu_1 + \int fd\nu_2$ for any ν_1 and ν_2 in V_n. Therefore, (31) implies that:

$$\int_\Omega fd\nu = \int_\Omega fd\left(\sum_{A\in\Sigma'} \alpha_A^\nu u_A\right) = \sum_{A\in\Sigma'} \alpha_A^\nu \int_\Omega fdu_A \tag{34}$$

for each $f \in B(\Sigma)$. Define a function $\widetilde{f} : \Sigma' \to \mathbb{R}$ by

$$\widetilde{f}(A) = \int_\Omega fdu_A \qquad \text{for each } A \in \Sigma'. \tag{35}$$

As $\int fdu_A = \min_{\omega\in A} f(\omega)$ (see Example 20), it actually holds

$$\widetilde{f}(A) = \min_{\omega\in A} f(\omega) \qquad \text{for each } A \in \Sigma'.$$

By (34) we have

$$\int_\Omega fd\nu = \sum_{A\in\Sigma'} \alpha_A^\nu \widetilde{f}(A) = \sum_{A\in\Sigma'} \alpha_A^\nu \min_{\omega\in A} f(\omega) = \int_{\Sigma'} \widetilde{f}d\mu_\nu,$$

and so the representation (34) can be written as:

$$\int_\Omega fd\nu = \int_{\Sigma'} \widetilde{f}d\mu_\nu \qquad \text{for each } f \in B(\Sigma). \tag{36}$$

This is the desired extension of (32) to all functions in $B(\Sigma)$. In fact, if $f = 1_E$, we have $\widetilde{f} = 1_{\widetilde{E}}$, and so (36) reduces to (32) for characteristic functions.

Summing up, the additive representation of the Choquet integral $\int fd\nu$ is given by

$$\int_{\Sigma'} \widetilde{f}d\mu_\nu = \int_{\Sigma'} \min_{\omega\in A} f(\omega)\, d\mu_\nu = \sum_{A\in\Sigma'} \alpha_A^\nu \min_{\omega\in A} f(\omega).$$

Theorem 53 can be extended from games to Choquet integrals along these lines. In order to do so, consider the space V_n^c of all Choquet functionals on

$B(\Sigma)$. It is a vector space since $(\alpha\nu' + \beta\nu'')_c = \alpha\nu'_c + \beta\nu''_c$ for all $\nu', \nu'' \in V_n$ and all $\alpha, \beta \in \mathbb{R}$. By the next result, V_n^c is isomorphic to the dual space $B(2^{\Sigma'})^*$ of $B(2^{\Sigma'})$.[17]

Corollary 54 *There is an isomorphism between the vector spaces V_n^c and $B(2^{\Sigma'})^*$ determined by the identity*

$$\nu_c(f) = \mu\left(\widetilde{f}\right) \qquad \text{for each } f \in B(\Sigma). \tag{37}$$

In particular, $I(\nu) = \mu$, where ν is the game associated to ν_c and I is the isomorphism of Theorem 53.

Remark. For convenience, here μ denotes the linear functional in $B(2^{\Sigma'})^*$ given by $\int f d\mu$ for each $f \in B(2^{\Sigma'})$.

Proof. We first show that given any $\mu \in B(2^{\Sigma'})^*$, the functional $\nu_c : B(\Sigma) \to \mathbb{R}$ defined by (37) is comonotonic additive. Observe that, given any two comonotonic f_1 and f_2 in $B(\Sigma)$, it holds:

$$\begin{aligned} \left(\widetilde{f_1 + f_2}\right)(A) &= \min_{\omega \in A} (f_1 + f_2)(\omega) \\ &= \min_{\omega \in A} f_1(\omega) + \min_{\omega \in A} f_2(\omega) = \widetilde{f}_1(A) + \widetilde{f}_2(A) \end{aligned} \tag{38}$$

for each $A \in \Sigma'$. Hence,

$$\begin{aligned} \nu_c(f_1 + f_2) &= \mu\left(\widetilde{f_1 + f_2}\right) = \mu\left(\widetilde{f}_1 + \widetilde{f}_2\right) \\ &= \mu\left(\widetilde{f}_1\right) + \mu\left(\widetilde{f}_2\right) = \nu_c(f_1) + \nu_c(f_2), \end{aligned}$$

and so ν_c is comonotonic additive, as desired.

It remains to prove that, given any Choquet functional $\nu_c \in V_n^c$, the functional μ defined by (37) is linear on $B(2^{\Sigma'})$. By Theorem 53, Eq. (37) uniquely determines a charge μ on the power set $2^{\Sigma'}$. Hence, the associated linear functional $\int f d\mu$ belongs to $B(2^{\Sigma'})^*$, as desired. ■

---

[17] $B\left(2^{\Sigma'}\right)^*$ is the vector space of all linear functionals defined on the vector space $B\left(2^{\Sigma'}\right)$ of all functions defined on the enlarged space Σ'.

6.4 Polynomial Representation

A further possible way to represent finite games is in terms of polynomials. Consider the set $\{0,1\}^n$ of the vertices of the hypercube $[0,1]^n$. Functions $f:\{0,1\}^n \to \mathbb{R}$ are called pseudo-Boolean (see [8] and [28]).

Say that a pseudo-Boolean function f is *grounded* if $f(0,...,0)=0$. Finite games can be regarded as grounded pseudo-Boolean functions. In fact, w.l.o.g. set $\Omega=\{1,...,n\}$ and $\Sigma=2^{\{1,...,n\}}$, so that V_n is the set of all games $\nu: 2^{\{1,...,n\}} \to \mathbb{R}$. Given $A \subseteq \{1,...,n\}$, consider the characteristic vector $1_A \in \{0,1\}^n$ given by:

$$1_A(i) = \begin{cases} 1 & i \in A \\ 0 & \text{else} \end{cases}$$

Since $\{0,1\}^n = \{1_A : A \subseteq \{1,...,n\}\}$, each game ν uniquely determines a grounded pseudo-Boolean function f by setting $f(1_A)=\nu(A)$ for each $A \subseteq \{1,...,n\}$. Conversely, each grounded pseudo-Boolean function f induces a game $\nu: 2^{\{1,...,n\}} \to \mathbb{R}$ by setting $\nu(A)=f(1_A)$ for each $A \subseteq \{1,...,n\}$.

Given a pseudo-Boolean function f, consider the polynomial

$$B_f(x) = \sum_{A \subseteq \{1,...,n\}} f(1_A) \prod_{i \in A} x_i \prod_{j \in A^c} (1-x_j) \quad \text{for each } x \in \mathbb{R}^n. \tag{39}$$

This polynomial is an extension of f on $\mathbb{R}^n$ as $B_f(1_A)=f(1_A)$ for each $A \subseteq \{1,...,n\}$. More important, B_f is a Bernstein polynomial of f. For, recall (see [59]) that given a function $f:[0,1]^n \to \mathbb{R}$ and an n-tuple $m=(m_1,m_2,...,m_n)$ with non-negative integer components, its Bernstein polynomial $B^m f: \mathbb{R}^n \to \mathbb{R}$ is

$$B^m f(x) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} \cdots \sum_{k_n=0}^{m_n} f\left(\frac{k_1}{m_1},...,\frac{k_n}{m_n}\right) \prod_{i=1}^{n} \binom{m_i}{k_i} x_i^{k_i} (1-x_i)^{m_i-k_i}.$$

In particular, the least-degree Bernstein polynomial $B^{(1,...,1)}f:\mathbb{R}^n \to \mathbb{R}$ associated with f is given by

$$B^{(1,...,1)}f(x) = \sum_{k=(k_1,...,k_n)\in\{0,1\}^n} f(k)\, x_1^{k_1} \cdots x_n^{k_n} (1-x_1)^{1-k_1} \cdots (1-x_n)^{1-k_n}.$$

To define $B^{(1,...,1)}$ we only need to know the values of f at the vertices $\{0,1\}^n$, and this makes it possible to associate $B^{(1,...,1)}$ to any pseudo-Boolean function. The polynomial (39) is, therefore, the least-degree Bernstein polynomial $B^{(1,...,1)}$ of $f:\{0,1\}^n \to \mathbb{R}$.

When f is grounded, the polynomial B_f is multilinear, that is, it is linear in each variable x_i. In particular, B_f is the unique multilinear extension of f on $\mathbb{R}^n$ and it can be also written as

$$B_\nu(x) = \sum_{\varnothing \neq A \in \Sigma} \nu(A) \prod_{i \in A} x_i \prod_{j \in A^c} (1 - x_j) \quad \text{for each } x \in \mathbb{R}^n, \tag{40}$$

where ν is the game associated with the grounded function f. The polynomial B_ν is called the Owen multilinear extension of the game ν, and it was introduced by Owen [47]. In view of our previous discussion, B_ν is the least-degree Bernstein polynomial of the grounded pseudo-Boolean function induced by the game.

Example 55 Consider the game $\nu \in V_n$ given by $\nu(A) = |A|^2$ for each $A \subseteq \{1, ..., n\}$. We have

$$B_\nu(x) = \sum_{i=1}^n x_i + 2 \sum_{i \neq j} x_i x_j.$$

▲

Denote by $\mathcal{P}_n$ the vector space of all multilinear polynomials on $\mathbb{R}^n$. The next result provides two basis for $\mathcal{P}_n$ and a formula for the relative change of basis.

Theorem 56 *Monomials $\prod_{i \in A} x_i$ form a basis for the $(2^n - 1)$-dimensional vector space $\mathcal{P}_n$, as well as the polynomials $\prod_{i \in A} x_i \prod_{i \in A^c} (1 - x_i)$. Given $P \in \mathcal{P}_n$, if*

$$P(x) = \sum_{\varnothing \neq A \in \Sigma} \alpha_A \prod_{i \in A} x_i = \sum_{\varnothing \neq A \in \Sigma} \beta_A \prod_{i \in A} x_i \prod_{i \in A^c} (1 - x_i),$$

then

$$\alpha_A = \sum_{B \subseteq A} (-1)^{|A| - |B|} \beta_B. \tag{41}$$

Proof. Each multilinear polynomial P can written as a linear combination $\sum_{\varnothing \neq A \in \Sigma} \alpha_A \prod_{i \in A} x_i$ of monomials. Let us prove that such combination is unique. As in [8], we proceed by induction on the size of the subsets A.

Begin with $|A| = 1$. In this case $\alpha_A = P(1_A)$, and so the coefficient α_A is uniquely determined.

Assume next that all α_A, with $|A| \leq k-1$, are uniquely determined. Let A be such that $|A| = k$. Since $P(1_A) = \sum_{B \subseteq A} \alpha_B$, we have

$$\alpha_A = P(1_A) - \sum_{B \subsetneq A} \alpha_B.$$

The coefficient α_A is then uniquely determined as all coefficients α_B are uniquely determined by the induction hypothesis. We conclude that the monomials are a basis for $\mathcal{P}_n$. As there are $2^n - 1$ monomials, the space $\mathcal{P}_n$ has dimension $2^n - 1$.

There are $2^n - 1$ polynomials of the form $\prod_{i \in A} x_i \prod_{i \in A^c} (1 - x_i)$. Hence, they form a basis provided they are linearly independent. To see that this the case, suppose

$$P(x) \equiv \sum_{\varnothing \neq A \in \Sigma} \beta_A \prod_{i \in A} x_i \prod_{i \in A^c} (1 - x_i) = 0 \quad \text{for each } x \in \mathbb{R}^n.$$

Then $P(1_A) = 0$ for each A, and so $\beta_A = 0$. This shows that these polynomials are linearly independent, and so a basis.

It remains to prove (41). Since $P(1_A) = \sum_{B \subseteq A} \alpha_B$ for each $A \subseteq \{1, ..., n\}$, we can obtain (41) by using a combinatorial argument that can be found in [60, p. 48] and [10, Lemma 2.3]. ■

Remark. Consider the function $\mathcal{M} : \mathcal{P}_n \to \mathcal{P}_n$ given by

$$\mathcal{M}(P)(1_A) = \sum_{B \subseteq A} (-1)^{|A| - |B|} P(1_B) \tag{42}$$

for each index set $A \subseteq \{1, ..., n\}$. This is the Mobius transform on $\mathcal{P}_n$ and, by (41), it can be viewed as a change of basis formula.

By Theorem 56, the polynomials $\prod_{i \in A} x_i \prod_{i \in A^c} (1 - x_i)$ form a basis for $\mathcal{P}_n$ and so each multilinear polynomial can be represented as in (39) and viewed as the least-degree Bernstein polynomial of a suitable grounded pseudo-Boolean function. Equivalently, each multilinear polynomial can be viewed as the Owen polynomial of a suitable game.

Moreover, by Theorem 56 we can represent the polynomial B_f of a grounded f in a unique way as

$$B_f(x) = \sum_{\varnothing \neq A \in \Sigma} \left(\sum_{B \subseteq A} (-1)^{|A|-|B|} f(1_B) \right) \prod_{i \in A} x. \tag{43}$$

Hence, the relative Owen polynomial can be uniquely written as

$$B_\nu(x) = \sum_{\varnothing \neq A \in \Sigma} \left(\sum_{B \subseteq A} (-1)^{|A|-|B|} \nu(B) \right) \prod_{i \in A} x.$$

Let us get back to finite games. Denote by B the Owen correspondence $\nu \mapsto B_\nu$ between V_n and $\mathcal{P}_n$. The next lemma collects a few simple properties of B. Here e_A denotes the game in V_n given by

$$e_A(B) = \begin{cases} 1 & A = B \\ 0 & \text{else} \end{cases}.$$

The family $\{e_A\}_{\varnothing \neq A \in \Sigma}$ is clearly a basis in V_n, and any game ν can be represented by $\nu = \sum_{\varnothing \neq A \in \Sigma} \nu(A) e_A$.

Lemma 57 *The Owen correspondence B is an isomorphism between the vector spaces V_n and $\mathcal{P}_n$. Moreover:*

(i) for each unanimity game u_A, we have

$$B_{u_A}(x) = \prod_{i \in A} x_i \quad \textit{for each } x \in \mathbb{R}^n;$$

(ii) for each game e_A, we have

$$B_{e_A}(x) = \prod_{i \in A} x_i \prod_{i \in A^c} (1 - x_i) \quad \textit{for each } x \in \mathbb{R}^n;$$

(iii) for each charge μ, we have

$$B_\mu(x) = \sum_{i=1}^{n} \mu(i) x_i \quad \textit{for each } x \in \mathbb{R}^n;$$

(iv) a game ν is positive if and only if $B_\nu(x) \geq 0$ for all $x \in [0,1]^n$.

(v) a game ν is convex if, for each $i \neq j$,

$$\frac{\partial B_\nu(x)}{\partial x_i \partial x_j} \geq 0 \quad \textit{for all } x \in (0,1)^n .$$

Proof. By Theorem 56, B is one-to-one. As it is also linear, B is an isomorphism between the vector spaces V_n and $\mathcal{P}_n$. Let us prove (i). As $\prod_{i \in A} x_i \in \mathcal{P}_n(x)$ and B is a linear isomorphism, there exists a unique game ν such that $B_\nu(x) = \prod_{i \in A} x_i$. As $\nu(B) = B_\nu(1_B)$, we have $\nu(B) = 1$ if $B \supseteq A$ and $\nu(B) = 0$ elsewhere. Hence, $\nu = u_A$.

As (ii) is trivially true, let us prove (iii). By Example 42, $\mu = \sum_{i=1}^n \mu(i)\,\delta_i$, where δ_i is the Dirac charge concentrated on $i \in \Omega$. By the linearity of B and by point (i),

$$B_\mu(x) = B_{\sum_{i=1}^n \mu(i)\delta_i}(x) = \sum_{i=1}^n \mu(i)\,B_{\delta_i}(x) = \sum_{i=1}^n \mu(i)\,x_i,$$

as desired.

(iv). If $\nu \geq 0$, the Owen polynomial (40) has all positive coefficients. As $\prod_{i \in A} x_i \prod_{j \in A^c}(1 - x_j) \geq 0$ on $[0,1]^n$, we then have $B_\nu(x) \geq 0$ on $[0,1]^n$. The converse is obvious as $\nu(A) = B_\nu(1_A) \geq 0$.

(v). This condition on the second derivatives implies that B_ν is supermodular on $(0,1)^n$. As it is continuous, B_ν is then supermodular on the hypercube $[0,1]^n$. In turn this implies the convexity of ν. ■

Lemma 57(i) shows that unanimity games are the game counterpart of monomials. By Theorem 56, monomials form a basis of the space $\mathcal{P}_n$ of multilinear polynomials. As a result, Theorem 41 can be viewed as a corollary of Theorem 56, and the representation (21) as a consequence of the polynomial representation (43).

Remark. As we did in $\mathcal{P}_n$ with (42), here as well we can define a Mobius transform $\mathcal{M} : V_n \to V_n$ by

$$\mathcal{M}(\nu)(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|}\,\nu(B)$$

for each $A \subseteq \{1, ..., n\}$. The Mobius transform on V_n can be viewed as a change of basis formula, between the basis $\{e_A\}_{\varnothing \neq A \in \Sigma}$ and $\{u_A\}_{\varnothing \neq A \in \Sigma}$.

The next result completes Lemma 57 by showing what is the polynomial counterpart of total monotonicity.

Lemma 58 *A game ν is totally monotone if and only if its Owen polynomial B_ν is nonnegative on $\mathbb{R}^n_+$, i.e., $B_\nu(x) \geq 0$ for each $x \in \mathbb{R}^n_+$.*

Proof. Suppose ν is totally monotone. By Lemma 57, we can write

$$B_\nu(x) = \sum_{\varnothing \neq A \in \Sigma} \alpha^\nu_A B_{u_A}(x) = \sum_{\varnothing \neq A \in \Sigma} \alpha^\nu_A \prod_{i \in A} x_i.$$

Hence, if ν is totally monotone, then $B_\nu(x) \geq 0$ for all $x \in \mathbb{R}^n_+$. Conversely, assume $B_\nu(x) \geq 0$ for all $x \in \mathbb{R}^n_+$. We want to show that ν is totally monotone, i.e., $\alpha^\nu_A \geq 0$ for each A. Suppose, *per contra*, that $\alpha^\nu_A < 0$ for some A. Consider the vector $t1_A \in \mathbb{R}^n_+$, with $t > 0$. Then,

$$B_\nu(t1_A) = \alpha^\nu_A t^{|A|} + \text{terms of lower degree}.$$

Hence, for t large enough we have $B_\nu(t1_A) < 0$, a contradiction. We conclude that $\alpha^\nu_A \geq 0$ for each A, as desired. ■

This lemma is the reason why we considered multilinear polynomials defined on $\mathbb{R}^n$ rather than on $[0,1]^n$, as it is usually the case. In fact, by Lemma 57(iv) the positivity of the Owen polynomial on $[0,1]^n$ only reflects the positivity of the associated game, not its total monotonicity.

We now illustrate Lemmas 57 and 58 with couple of examples.

Example 59 Consider the game $\nu(A) = |A|^2$ of Example 55. As

$$B_\nu(x) = \sum_{i=1}^n x_i + 2\sum_{i \neq j} x_i x_j \geq 0 \quad \text{for each } x \in \mathbb{R}^n_+,$$

by Lemma 58 the game ν is totally monotone. ▲

Example 60 Consider the game associated with the multilinear polynomial

$$B(x) = x_1x_2 + x_1x_3 + x_2x_3 - \varepsilon x_1x_2x_3$$

with $\varepsilon > 0$. As $B(10/\varepsilon, 10/\varepsilon, 2/\varepsilon) < 0$ for each $\varepsilon > 0$, this game is not totally monotone. The game is positive and convex when $\varepsilon \leq 1$. In fact,

$$B(x) = x_1 x_2 (1 - \varepsilon x_3) + x_1 x_3 + x_2 x_3 \geq 0$$

on $[0,1]^3$, and so by Lemma 57(iv) is positive. On the other hand,

$$\frac{\partial^2 B}{\partial x_i \partial x_j} = 1 - \varepsilon x_k \geq 0,$$

on $(0,1)^n$, so that, by Lemma 57(v), the game is convex. ▲

In view of Lemma 58 is natural to consider the pointed convex cone $\mathcal{P}_n^+ = \{P \in \mathcal{P}_n : P(x) \geq 0 \text{ for each } x \in \mathbb{R}_+^n\}$. It induces in the usual way an order $\succeq_p$ on $\mathcal{P}_n$ as follows: given $P_1, P_2 \in \mathcal{P}_n$, write $P_1 \succeq_p P_2$ if $P_1 - P_2 \in \mathcal{P}_n^+$. In turn, $\succeq_p$ induces a lattice structure and norm, denoted by $\|\cdot\|_p$, that makes $\mathcal{P}_n$ an AL-space. For brevity, we omit the details of these by now standard notions.

The next result summarizes the relations existing between the space of finite games and the space of multilinear polynomials just introduced.

Theorem 61 *There is a lattice preserving and isometric isomorphism B between the AL-spaces $(V_n, \succeq, \|\cdot\|)$ and $\left(\mathcal{P}_n, \succeq_p, \|\cdot\|_p\right)$ determined by the identity*

$$P(x) = \sum_{\varnothing \neq A \in \Sigma} \nu(A) \prod_{i \in A} x_i \prod_{j \in A^c} (1 - x_j) \quad \text{for each } x \in \mathbb{R}^n.$$

The game ν is totally monotone if and only if the corresponding polynomial P in $\mathcal{P}_n$ is nonnegative on $\mathbb{R}_+^n$.

Summing up, Theorems 50, 53, and 61 established the following lattice isometries:

$$\begin{array}{ccccc} \left(\mathbb{R}^{2^{|\Sigma|}-1}, \geq, \|\cdot\|_1\right) & \overset{T}{\longleftrightarrow} & (V_n, \succeq, \|\cdot\|) & \overset{I}{\longleftrightarrow} & \left(ba\left(2^{\Sigma'}\right), \succeq_{ba}, \|\cdot\|\right) \\ & & \updownarrow B & & \\ & & \left(\mathcal{P}_n, \succeq_p, \|\cdot\|_p\right) & & \end{array}$$

The resulting isometries $I \circ T^{-1}$ and $B \circ T^{-1}$ between $\mathbb{R}^{2^{|\Sigma|}-1}$, $ba\left(2^{\Sigma'}\right)$, and $\mathcal{P}_n$ are obviously well-known. The interesting part here is given by the possibility of representing finite games in different ways, each useful for different purposes.

6.5 Convex Games

In this last subsection we show some noteworthy properties of finite convex games. A first important property has been already mentioned right after Theorem 52: any finite game can be written as the difference of two convex games.

To see other properties of finite convex games, we have to turn our attentions to chains of subsets of Ω. As $\Omega = \{\omega_1, ..., \omega_n\}$, the collection $\mathcal{C}$ given by

$$\{\omega_1\}, \{\omega_1, \omega_2\}, ..., \{\omega_1, ..., \omega_n\}$$

forms a maximal chain, that is, no other chain can contain it. More generally, given any permutation σ on $\{1, ..., n\}$, the collection $\mathcal{C}_\sigma$ given by

$$\left\{\omega_{\sigma(1)}\right\}, \left\{\omega_{\sigma(1)}, \omega_{\sigma(2)}\right\}, ..., \left\{\omega_{\sigma(1)}, ..., \omega_{\sigma(n)}\right\}$$

forms another maximal chain. All maximal chains in Ω have this form, and so there are $n!$ of them.

Let ν be any game. By Lemma 28, for each $\mathcal{C}_\sigma$ there is a charge $\mu_\sigma \in ba(\Sigma)$ such that $\mu_\sigma(A) = \nu(A)$ for each $A \in \mathcal{C}_\sigma$. Because of the maximality of $\mathcal{C}_\sigma$, the charge μ_σ is easily seen to be unique. We call μ_σ is the *marginal worth charge* associated with permutation σ.

Marginal worth charges play a central role in studying finite convex games. We begin by providing a characterization of convexity based on them, due to Ichiishi [33].

Theorem 62 *A finite game ν is convex if and only if all its marginal worth charges μ_σ belong to the core.*

Proof. "Only if". Suppose ν is convex. We want to show that each μ_σ belongs to $core(\nu)$. By Theorem 38, there exists $\mu \in core(\nu)$ such that $\mu(A) = \nu(A)$ for each $A \in \mathcal{C}_\sigma$. By the maximality of $\mathcal{C}_\sigma$, μ_σ is the unique charge having such property. Hence, $\mu = \mu_\sigma$, as desired.

"If". Suppose $\mu_\sigma \in core(\nu)$ for all permutations σ. Given any A and B, let $\mathcal{C}_\sigma$ be a maximal chain containing $A \cap B$, A, and $A \cup B$. Then:

$$\nu(A \cup B) + \nu(A \cap B) - \nu(A) = \mu_\sigma(A \cup B) + \mu_\sigma(A \cap B) - \mu_\sigma(A) = \mu_\sigma(B).$$

As $\mu_\sigma \in core(\nu)$, we have $\mu_\sigma(B) \geq \nu(B)$, and so ν is convex. ■

Turn now to cores of finite games. The first observation to make is that the core of a finite game is a subset of the $|\Omega|$-dimensional space $\mathbb{R}^{\Omega}$ of the form:

$$core(\nu) = \left\{ x \in \mathbb{R}^{\Omega} : \sum_{\omega \in \Omega} x_{\omega} = \nu(\Omega) \text{ and } \sum_{\omega \in A} x_{\omega} \geq \nu(A) \text{ for each } A \right\}.$$

Equivalently,

$$core(\nu) = \bigcap_{A \in \Sigma} \left\{ x \in \mathbb{R}^{\Omega} : \sum_{\omega \in A} x_{\omega} \geq \nu(A) \right\} \cap \left\{ x \in \mathbb{R}^{\Omega} : \sum_{\omega \in \Omega} x_{\omega} \leq \nu(\Omega) \right\},$$

that is, $core(\nu)$ is the set of solutions of a finite system of linear inequalities on $\mathbb{R}^{\Omega}$. Sets of this form are called polyhedra.

By Proposition 3 the core is weak*-compact. In this finite setting, this means that it is a compact subset of $\mathbb{R}^{\Omega}$, where compactness is in the standard norm topology of $\mathbb{R}^{\Omega}$. The core of a finite game is, therefore, a compact polyhedron. As a result, we have the following geometric property of cores of finite games.

Proposition 63 *The core of a finite game is a polytope in* $\mathbb{R}^{\Omega}$, *that is, it is the convex hull of a finite set.*

Proof. By a standard result (see [1, pp. 233-234] or [68, p. 114]), compact polyhedra are polytopes. ■

The extreme points of a polytope are called vertices and they form a finite set. As each element of a polytope can be represented as a convex combination of its vertices, the knowledge of the set of vertices is, therefore, key in describing the structure of a polytope.

All this means that, by Proposition 63, in order to understand the structure of the core it is crucial to identify the set of its vertices. This is achieved by the next result, due to Shapley [63]. Interestingly, the marginal worth charges, which by Theorem 62 always belong to the core of a convex game, turn out to be exactly the sought-after vertices.

Theorem 64 *Let* ν *be a finite convex game. Then, a charge* $\mu \in ba(\Sigma)$ *is a vertex of* $core(\nu)$ *if and only if it is a marginal worth charge, that is, if and only if there is a maximal chain* $\mathcal{C}_{\sigma}$ *such that* $\nu(A) = \mu(A)$ *for all* $A \in \mathcal{C}_{\sigma}$.

Proof. An element of a polytope is a vertex if and only if it is an exposed point. Hence, it is enough to show that the marginal worth charges are the set of exposed points of *core* (ν).

"If". Suppose μ_σ is a marginal worth charge, with associated maximal chain $\mathcal{C}_\sigma$. We want to show that is an exposed point of *core* (ν). Since $\mathcal{C}_\sigma$ is a maximal chain, there is an injective function f_σ whose upper sets are given by $\mathcal{C}_\sigma$, i.e., $\mathcal{C}_\sigma = \{(f_\sigma \geq t)\}_{t\in\mathbb{R}}$. For example, if $\mathcal{C}_\sigma = \left\{A_{\sigma(i)}\right\}$, take $f_\sigma = \sum_{i=1}^n 1_{A_{\sigma(i)}}$. By the definition of Choquet integral, we have $\int f d\mu_\sigma = \int f d\nu$. Since $\mathcal{C}_\sigma$ is maximal, μ_σ is the unique charge replicating ν on $\mathcal{C}_\sigma$. Therefore, given any other charge μ in *core* (ν), there exists $A \in \mathcal{C}_\sigma$ such that $\mu_\sigma(A) < \mu(A)$. Equivalently, there is some $t \in \mathbb{R}$ such that $\nu(f_\sigma \geq t) = \mu_\sigma(f_\sigma \geq t) < \mu(f_\sigma \geq t)$. Hence, $\int f_\sigma d\nu = \int f_\sigma d\mu_\sigma < \int f_\sigma d\mu$ for all $\mu \in$ *core* (ν) with $\mu \neq \mu_\sigma$, and this proves that μ_σ is an exposed point, as desired.

"Only if". Suppose μ^* is an exposed point of *core* (ν). We want to show that μ^* is a marginal worth charge, i.e., that there exists a maximal chain $\mathcal{C}^*$ in Ω such that $\mu^*(A) = \nu(A)$ for each $A \in \mathcal{C}^*$.

Let $\{\mu_i\}_{i=1}^m$ be the set of all exposed points of *core* (ν), except μ^*. Set $k_1 = \|\mu^*\| \vee (\max_{i=1,\ldots,m}\|\mu_i\|)$. Since μ^* is an exposed point, there exists $f : \Omega \to \mathbb{R}$ such that $\int f d\mu^* < \int f d\mu$ for all $\mu \in$ *core* (ν) with $\mu \neq \mu^*$. Set $k_2 = \min_{i=1,\ldots,m}\left(\int f d\mu_i - \int f d\mu^*\right)$. Clearly, $k_2 > 0$. Given $0 < \varepsilon < k_2/2k_1$, there is an injective $g : \Omega \to \mathbb{R}$ such that $\|f - g\| < \varepsilon$. Hence, for each i we have:

$$\begin{aligned}\int g d\mu_i - \int g d\mu^* &= \int g d\mu_i - \int f d\mu_i + \int f d\mu_i - \int f d\mu^* \\ + \int f d\mu^* - \int g d\mu^* &\geq -\varepsilon k_1 + k_2 - \varepsilon k_1 > 0.\end{aligned}$$

We conclude that $\int g d\mu^* < \int g d\mu_i$ for each i, and so $\int g d\mu^* < \int g d\mu$ for all $\mu \in$ *core* (ν) with $\mu \neq \mu^*$.

Since ν is convex, by Theorem 38 it holds $\int g d\nu = \min_{\mu\in core(\nu)} \int g d\mu$, and so $\int g d\nu = \int g d\mu^* < \int g d\mu$ for all $\mu \in$ *core* (ν) with $\mu \neq \mu^*$. The equality $\int g d\nu = \int g d\mu^*$ implies that $\mu^*(g \geq t) = \nu(g \geq t)$ for all $t \in \mathbb{R}$. Since g is injective, the chain of upper sets $\{g \geq t\}$ is maximal in Ω, and it is actually the desired maximal chain $\mathcal{C}^*$. ■

Denote by $M(\nu)$ the set of all marginal worth charges of a game ν. By Theorem 64, we have *core* $(\nu) = co(M(\nu))$, and so all elements of the core can be represented as convex combinations of marginal worth charges.

This result has been recently generalized to infinite games by Marinacci and Montrucchio [43].

Putting together Theorems 62 and 64, we have the following remarkable property of finite games.

Corollary 65 *A finite game ν is convex if and only if*

$$M(\nu) = \exp(core(\nu)).$$

Therefore, given a game, the knowledge of its $n!$ marginal worth charges makes it possible to determine both whether the game is convex and what is the structure of its core.

We close by observing that it is not by chance that in Corollary 65 we use the set of exposed point *exp* rather than that of extreme points *ext*. For a polytope these two sets coincide and they form the set of vertices. For general compact convex sets, even in finite dimensional spaces, this is no longer the case and exposed points are only a subset of the set of extreme points. Inspection of the proof of Theorem 64 shows that what we have actually proved is that marginal worth charges are the set of exposed points of the core. The fact that they then turn out to coincide with the set of extreme points is a consequence of properties of polytopes, which are immaterial for the proof.

When extending the result to infinite convex games this observation is important as in the more general setting – where exposed and extreme points no longer necessarily coincide – the analog of marginal worth charges will actually characterize the exposed points. We refer the interested reader to [43] for details.

7 Concluding Remarks

1. In this chapter we only considered games defined on spaces having no topological structure. There is a large literature on suitably "regular" set functions defined on topological spaces, tracing back to Choquet [11]. We refer the interested reader to Huber and Strassen [32] and Dellacherie and Meyer [14]. Epstein and Wang [22] and Philippe, Debs, and Jaffray [49] provide some decision-theoretic applications of capacities on topological domains.

2. In a series of papers, Gabriele Greco proposed an interesting notion of measurability on algebras. A noteworthy feature of his approach is that, unlike $B(\Sigma)$, the resulting class of measurable functions forms a vector space. Greco's approach is, therefore, a further way to bypass the lack of vector structure of $B(\Sigma)$ that we discussed in some detail after Theorem 35. In this chapter, we preferred to define the Choquet functional on the smaller domain $B(\Sigma)$ and then extend it on the vector space $\overline{B}(\Sigma)$ using its Lipschitz continuity, following in this way a standard procedure in functional analysis.

In any case, details on Greco's approach can be found in his papers (e.g., [30] and [3]) and in Denneberg [17].

3. We did not consider here games and Choquet functionals defined on product algebras. For details on this topic we refer the interested reader to Ben Porath, Gilboa, and Schmeidler [4], Ghirardato [24], and to the references therein contained.

4. Throughout the chapter we only considered Choquet functionals defined on bounded functions. Results for the unbounded case can be found in Greco [29], [31], and [3], and in Wakker [67].

5. Sipos [65] and [66] introduced a different notion of integral for capacities. It coincides with the Choquet integral for positive functions, but the extension to general functions is done according to the standard procedure used to extend the Lebesgue integral from positive functions to general functions, based on the decomposition $f = f^+ - f^-$. The resulting integral is in general different from the Choquet integral and it turned out to be useful in some applications. We refer the interested reader to Sipos' original papers and to Denneberg [17].

6. Theorem 35 and Corollary 37 make it possible to use convex analysis tools in studying convex games and their Choquet integrals. For example, Carlier and Dana [9] and Marinacci and Montrucchio [43] use such tools to study the structure of cores of convex games and the differentiability and subdifferentiability properties of their Choquet integrals.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, Springer-Verlag, New York, 1999.

[2] R. Aumann and L. Shapley, *Values of non-atomic games*, Princeton University Press, Princeton, 1974.

[3] R. C. Bassanezi and G. H. Greco, Sull'additività dell'integrale, *Rendiconti Seminario Matematico Università di Padova*, 72, 249-275, 1984.

[4] E. Ben Porath, I. Gilboa, and D. Schmeidler, On the measurement of inequality under uncertainty, *Journal of Economic Theory*, 75, 194-204, 1997.

[5] K.P.S. Bhaskara Rao, M. Bhaskara Rao, *Theory of charges*, Academic Press, New York, 1983.

[6] A.K. Biswas, T. Parthasarathy, J.A.M. Potters and M. Voorneveld, Large cores and exactness, *Games and Economic Behavior,* 28, 1-12, 1999.

[7] O. Bondareva, Certain applications of the methods of linear programming to the theory of cooperative games, (in Russian) *Problemy Kibernetiki*, 10, 119-139, 1963.

[8] E. Boros and P. L. Hammer, Pseudo-Boolean optimization, *Discrete Applied Mathematics*, 123, 155-225, 2002.

[9] G. Carlier and R. A. Dana, Core of convex distortions of a probability on a non-atomic space, *Journal of Economic Theory*, forthcoming.

[10] A. Chateauneuf and J.-Y. Jaffray, Some characterizations of lower probabilities and other monotone capacities through the use of Mobius inversion, *Mathematical Social Sciences* 17, 263-283, 1989.

[11] G. Choquet, Theory of capacities, *Annales de l'Institut Fourier*, 5, 131-295, 1953.

[12] F. Delbaen, Convex games and extreme points, *Journal of Mathematical Analysis and Applications*, 45, 210-233, 1974.

[13] C. Dellacherie, Quelques commentaires sur le prolongements de capacités, Seminaire Probabilités V, *Lecture Notes in Math.* 191, Springer-Verlag, New York, 1971.

[14] C. Dellacherie and P.-A. Meyer, *Probabilities and potential*, North-Holland, Amsterdam, 1978.

[15] A. Dempster, Upper and lower probabilities induced by a multivalued mapping, *Annals of Mathematical Statistics*, 38, 325-339, 1967.

[16] A. Dempster, A generalization of Bayesian inference, *Journal of the Royal Statistical Society (B),* 30, 205-247, 1968.

[17] D. Denneberg, *Non-additive measure and integral*, Kluwer, Dordrecht, 1994.

[18] D. Denneberg, Representation of the Choquet integral with the σ-additive Mobius transform, *Fuzzy Sets and Systems*, 92, 139-156, 1997.

[19] A. De Waegenaere and P. Wakker, Nonmonotonic Choquet integrals, *Journal of Mathematical Economics*, 36, 45-60, 2001.

[20] N. Dunford and J.T. Schwartz, *Linear operators, part I: general theory*, Wiley-Intersience, London, 1958.

[21] E. Einy and B. Shitovitz, Convex games and stable sets, *Games and Economic Behavior,* 16, 192-201, 1996.

[22] L. G. Epstein and T. Wang, "Beliefs about beliefs" without probabilities, *Econometrica*, 64, 1343-1373, 1996.

[23] K. Fan, On systems of linear inequalities, in Linear inequalities and related systems, *Annals of Math. Studies*, 38, 99-156, 1956.

[24] P. Ghirardato, On independence for non-additive measures, with a Fubini theorem, *Journal of Economic Theory*, 73, 261–291, 1997.

[25] I. Gilboa and E. Lehrer, Global games, *International Journal of Game Theory,* 20, 129-147, 1991.

[26] I. Gilboa and D. Schmeidler, Additive representations of non-additive measures and the Choquet integral, *Annals of Operations Research*, 52, 43-65, 1994.

[27] I. Gilboa and D. Schmeidler, Canonical representation of set functions, *Mathematics of Operations Research,* 20, 197-212, 1995.

[28] M. Grabisch, J.-L. Marichal, and M. Roubens, Equivalent representations of set functions, *Mathematics of Operations Research*, 25, 157-178, 2000.

[29] G. H. Greco, Integrale monotono, *Rendiconti Seminario Matematico Università di Padova,* 57, 149-166, 1976.

[30] G. H. Greco, Sur la mesurabilité d'une fonction numérique par rapport à une famille d'ensembles. *Rendiconti Seminario Matematico Università di Padova,* 65, 163-176, 1981.

[31] G. H. Greco, Sulla rappresentazione di funzionali mediante integrali, *Rendiconti Seminario Matematico Università di Padova,* 66, 21-42, 1982.

[32] P. J. Huber and V. Strassen, "Minimax tests and the Neyman-Pearson lemma capacities," *Annals of Statistics* 1, 251-263, 1973.

[33] T. Ichiishi, Super-modularity: applications to convex games and to the greedy algorithm for LP, *Journal of Economic Theory*, 25, 283-286, 1981.

[34] Y. Kannai, Countably additive measures in cores of games, *Journal of Mathematical Analysis and Applications*, 27, 227-240, 1969.

[35] J. L. Kelley, Measures on Boolean algebras, *Pacific Journal of Mathematics*, 9, 1165–1177, 1959.

[36] K. Kikuta, A condition for a game to be convex, *Mathematica Japonica*, 33, 425-430, 1988.

[37] K. Kikuta and L. S. Shapley, Core stability in n-person games, mimeo, 1986.

[38] F. Maccheroni and M. Marinacci, An Heine-Borel theorem for $ba\,(\Sigma)$, mimeo.

[39] F. Maccheroni and W. H. Ruckle, BV as a dual space, *Rendiconti Seminario Matematico Università di Padova*, 107, 101-109, 2002.

[40] M. Marinacci, Decomposition and representation of coalitional games, *Mathematics of Operations Research*, 21, 1000-1015, 1996.

[41] M. Marinacci, Finitely additive and epsilon Nash equilibria, *International Journal of Game Theory,* 26, 315-333, 1997.

[42] M. Marinacci and L. Montrucchio, Subcalculus for set functions and cores of TU games, *Journal of Mathematical Economics*, 39, 1-25, 2003.

[43] M. Marinacci and L. Montrucchio, A Characterization of the core of convex games through Gateaux derivatives, *Journal of Economic Theory,* forthcoming.

[44] H. Moulin, *Cooperative microeconomics*, Princeton Univ. Press, Princeton, 1995.

[45] R. Myerson, *Game theory*, Harvard Univ. Press, Cambridge, 1991.

[46] H. T. Nguyen, On random sets and belief functions, *Journal of Mathematical Analysis and Applications*, 65, 531-542, 1978.

[47] G. Owen, Multilinear extensions of games, *Management Science*, 18, 64-79, 1972.

[48] G. Owen, *Game theory*, Academic Press, New York, 1995.

[49] F. Philippe, G. Debs, and J-Y. Jaffray, Decision making with monotone lower probabilities of infinite order, *Mathematics of Operations Research*, 24, 767-784, 1999.

[50] A. Revuz, Fonctions croissantes et mesures sur les espaces topologiques ordonnés, *Annales de l'Institut Fourier*, 6, 187-269, 1955.

[51] G. C. Rota, Theory of Mobius functions, *Z. Wahrsch. und Verw. Geb.*, 2, 340-368, 1964.

[52] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

[53] W. Rudin, *Real and complex analysis* (3$^{\text{rd}}$ edition), McGraw-Hill, New York, 1987.

[54] G. Salinetti and R. Wets, On the convergence in distribution of measurable multifunctions (random sets), normal integrands, stochastic processes and stochastic infima, *Mathematics of Operations Research*, 11, 385–419, 1986.

[55] D. Schmeidler, On balanced games with infinitely many players, Research Program in Game Theory and Mathematical Economics, RM 28, The Hebrew University of Jerusalem, 1968.

[56] D. Schmeidler, Integral representation without additivity, *Proceedings of the American Mathematical Society*, 97, 255-261, 1986.

[57] D. Schmeidler, Cores of exact games, *Journal of Mathematical Analysis and Applications*, 40, 214-225, 1972.

[58] D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica*, 57, 571-587, 1989.

[59] M. H. Schultz, L^∞-multivariate approximation theory, *SIAM Journal on Numerical Analysis* 6, 161-183, 1969.

[60] G. Shafer, *A mathematical theory of evidence*, Princeton University Press, Princeton, 1976.

[61] L. S. Shapley, A value for n-person games, in *Contributions to the Theory of Games* (H. Kuhn and A. W. Tucker, eds.), Princeton Univ. Press, Princeton, 1953.

[62] L. S. Shapley, On balanced sets and cores, *Naval Research Logistic Quarterly*, 14, 453-460, 1967.

[63] L. S. Shapley, Cores of convex games, *International Journal of Game Theory*, 1, 12-26, 1971.

[64] W. W. Sharkey, Cooperative games with large cores, *International Journal of Game Theory,* 11, 175-182, 1982.

[65] J. Sipos, Integral with respect to a pre-measure, *Mathematica Slovaca*, 29, 141-155, 1979.

[66] J. Sipos, Non linear integrals, *Mathematica Slovaca,* 29, 257-270, 1979.

[67] P. Wakker, Unbounded utility for Savage's "foundations of statistics" and other models, *Mathematics of Operations Research*, 18, 446–485, 1993.

[68] R. Webster, *Convexity*, Oxford Univ. Press, Oxford, 1994.

[69] D. V. Widder, *The Laplace transform*, Princeton Univ. Press, 1941.

[70] M. Wolfenson and T. L. Fine, Bayes-like decision making with upper and lower probabilities, *Journal of the American Statistical Association*, 77, 80-88, 1982.

[71] L. Zhou. Integral representation of continuous comonotonically additive functionals, *Transactions of the American Mathematical Society* 350, 1811-1822, 1998.

INTERNATIONAL CENTRE FOR ECONOMIC RESEARCH
APPLIED MATHEMATICS WORKING PAPER SERIES

1. Luigi Montrucchio and Fabio Privileggi, "On Fragility of Bubbles in Equilibrium Asset Pricing Models of Lucas-Type," *Journal of Economic Theory* 101, 158-188, 2001 (ICER WP 2001/5).
2. Massimo Marinacci, "Probabilistic Sophistication and Multiple Priors," *Econometrica* 70, 755-764, 2002 (ICER WP 2001/8).
3. Massimo Marinacci and Luigi Montrucchio, "Subcalculus for Set Functions and Cores of TU Games," *Journal of Mathematical Economics* 39, 1-25, 2003 (ICER WP 2001/9).
4. Juan Dubra, Fabio Maccheroni, and Efe Ok, "Expected Utility Theory without the Completeness Axiom," *Journal of Economic Theory*, forthcoming (ICER WP 2001/11).
5. Adriana Castaldo and Massimo Marinacci, "Random Correspondences as Bundles of Random Variables," April 2001 (ICER WP 2001/12).
6. Paolo Ghirardato, Fabio Maccheroni, Massimo Marinacci, and Marciano Siniscalchi, "A Subjective Spin on Roulette Wheels," *Econometrica*, forthcoming (ICER WP 2001/17).
7. Domenico Menicucci, "Optimal Two-Object Auctions with Synergies," July 2001 (ICER WP 2001/18).
8. Paolo Ghirardato and Massimo Marinacci, "Risk, Ambiguity, and the Separation of Tastes and Beliefs," *Mathematics of Operations Research* 26, 864-890, 2001 (ICER WP 2001/21).
9. Andrea Roncoroni, "Change of Numeraire for Affine Arbitrage Pricing Models Driven By Multifactor Market Point Processes," September 2001 (ICER WP 2001/22).
10. Maitreesh Ghatak, Massimo Morelli, and Tomas Sjoström, "Credit Rationing, Wealth Inequality, and Allocation of Talent", September 2001 (ICER WP 2001/23).
11. Fabio Maccheroni and William H. Ruckle, "BV as a Dual Space," *Rendiconti del Seminario Matematico dell'Università di Padova*, 107, 101-109, 2002 (ICER WP 2001/29).
12. Fabio Maccheroni, "Yaari Dual Theory without the Completeness Axiom," *Economic Theory*, forthcoming (ICER WP 2001/30).
13. Umberto Cherubini and Elisa Luciano, "Multivariate Option Pricing with Copulas," January 2002 (ICER WP 2002/5).
14. Umberto Cherubini and Elisa Luciano, "Pricing Vulnerable Options with Copulas," January 2002 (ICER WP 2002/6).
15. Steven Haberman and Elena Vigna, "Optimal Investment Strategies and Risk Measures in Defined Contribution Pension Schemes," *Insurance: Mathematics and Economics* 31, 35-69, 2002 (ICER WP 2002/10).
16. Enrico Diecidue and Fabio Maccheroni, "Coherence without Additivity," *Journal of Mathematical Psychology*, forthcoming (ICER WP 2002/11).
17. Paolo Ghirardato, Fabio Maccheroni, and Massimo Marinacci, "Ambiguity from the Differential Viewpoint," April 2002 (ICER WP 2002/17).

18. Massimo Marinacci and Luigi Montrucchio, "A Characterization of the Core of Convex Games through Gateaux Derivatives," *Journal of Economic Theory*, forthcoming (ICER WP 2002/18).
19. Fabio Maccheroni and Massimo Marinacci, "How to Cut a Pizza Fairly: Fair Division with Decreasing Marginal Evaluations," *Social Choice and Welfare*, 20, 457-465, 2003 (ICER WP 2002/23).
20. Erio Castagnoli, Fabio Maccheroni and Massimo Marinacci, "Insurance Premia Consistent with the Market," *Insurance: Mathematics and Economics* 31, 267-284, 2002 (ICER WP 2002/24).
21. Fabio Privileggi and Guido Cozzi, "Wealth Polarization and Pulverization in Fractal Societies," September 2002 (ICER WP 2002/39).
22. Paolo Ghirardato, Fabio Maccheroni, and Massimo Marinacci, "Certainty Independence and the Separation of Utility and Beliefs," December 2002 (ICER WP 2002/40).
23. Salvatore Modica and Marco Scarsini, "The Convexity-Cone Approach to Comparative Risk and Downside Risk", January 2003 (ICER WP 2003/1).
24. Claudio Mattalia, "Existence of Solutions and Asset Pric ing Bubbles in General Equilibrium Models", January 2003 (ICER WP 2003/2).
25. Massimo Marinacci and Luigi Montrucchio, "Cores and Stable Sets of Finite Dimensional Games", March 2003 (ICER WP 2003/7).
26. Jerome Renault, Sergio Scarlatti, and Marco Scarsini, "A Folk Theorem for Minority Games", April 2003 (ICER WP 2003/10).
27. Peter Klibanoff, Massimo Marinacci, and Sujoy Mukerji, "A Smooth Model of Decision Making under Ambiguity", April 2003 (ICER WP 2003/11).
28. Massimo Marinacci and Luigi Montrucchio, "Ultramodular Functions", June 2003 (ICER WP 2003/13).
29. Erio Castagnoli, Fabio Maccheroni, and Massimo Marinacci, "Choquet Insurance Pricing: a Caveat", June 2003 (ICER WP 2003/14).
30. Thibault Gajdos and Eric Maurin, "Unequal Uncertainties and Uncertain Inequalities: an Axiomatic Approach, June 2003 (ICER WP 2003/15).
31. Thibault Gajdos and John A. Weymark, "Multidimensional Generalized Gini Indices", June 2003 (ICER WP 2003/16).
32. Thibault Gajdos, Jean-Marc Tallon, and Jean-Christophe Vergnaud, "Decision Making with Imprecise Probabilistic Information", June 2003 (ICER WP 2003/18).
33. Alfred Müller and Marco Scarsini, "Archimedean Copulae and Positive Dependence", July 2003 (ICER WP 2003/25).
34. Bruno Bassan, Olivier Gossner, Marco Scarsini, and Shmuel Zamir, "Positive Value of Information in Games", *International Journal of Game Theory*, forthcoming (ICER WP 2003/26).
35. Marco Dall'Aglio and Marco Scarsini, "Zonoids, Linear Dependence, and Size-Biased Distributions on the Simplex", *Advances in Applied Probability* forthcoming (ICER WP 2003/27).
36. Taizhong Hu, Alfred Muller, and Marco Scarsini, "Some Counterexamples in Positive Dependence", *Journal of Statistical Planning and Inference*, forthcoming (ICER WP 2003/28).
37. Massimo Marinacci, Fabio Maccheroni, Alain Chateauneuf, and Jean-Marc Tallon, "Monotone Continuous Multiple Priors" August 2003 (ICER WP 2003/30).